Virtues and Vices of Quartic Time–Frequency Distributions

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***Abstract*—We present results concerning three different types of quartic (fourth order) time–frequency distributions (TFDs). First, we present new results on the recently introduced *local ambiguity function* and show that it provides more reliable estimates of instantaneous chirp rate than the Wigner distribution. Second, we introduce the class of quartic, shift-covariant, time–frequency distributions and investigate distributions that localize quadratic chirps. Finally, we present a shift covariant distribution of time and chirp rate.**

***Index Terms*—Chirp modulation, estimation, signal representation, time–frequency analysis, Wigner distributions.**

I. Introduction

THE NOTION of a time–frequency distribution (TFD) [1]–[3] is inherently a concept that is not well defined [4]. A frequency is something that is measured over a period of time (e.g., how many times does the heart beat in a minute), and we would like to specify this frequency description at an instant of time (e.g., how fast is the heart beating right now). Nevertheless, TFDs have proven to be useful in many applications [5].

TFDs have been defined in a variety of ways. They can be a linear function of the signal like the short-time Fourier transform [1]–[3] and the continuous wavelet transform [2], [3], [6]. They can be quadratic functions of the signal like the Wigner distribution [1]–[3], the Cohen class [1],[1] hyperbolic distributions [7], and the Bertrand distributions [8]. Others have specific higher order forms such as the higher order Cohen classes [9], [10], the L-Wigner distributions [11], [12], and the polynomial Wigner–Ville distributions [13], [14]. Others do not fit into any of the above categories like the reassigned spectrogram [15], adaptive kernel distributions [16], and positive distributions [17].

Despite the abundance of methods for defining TFDs, there has yet to be an in-depth investigation of quartic (fourth-order) TFDs. In this paper, we will present results concerning three different types of quartic distributions. The first is the recently introduced *local ambiguity function*, the second is the class of quartic, shift-covariant TFDs, and the third is a distribution of time and chirp-rate. We show that these quartic methods provide results that are not obtainable with the simpler, linear or quadratic distributions.

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[1]We will use the term "Cohen class" to refer to the class of quadratic, shift-covariant TFDs [2], [3]. The Cohen class includes more general TFDs [1], but for the purposes of this paper, we prefer the more restrictive definition.

II. Local Ambiguity Function

Recently, a peculiar, quartic function called the *local ambiguity function* (LAF) has been derived [19]–[21] and presented in several, rather complicated forms [19]. The LAF combines elements of the Wigner distribution and the ambiguity function, which are defined as

$$W_x(t, \omega) = \int x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-j\omega\tau}\, d\tau$$
$$A_x(\theta, \tau) = \frac{1}{2\pi} \int x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{j\theta t}\, dt$$

respectively. One method for computing the LAF is

$$Q_x(t, \omega, \tau, \theta) = \frac{1}{2\pi} \iint W_x\left(t+\frac{\hat{\tau}}{2}, \omega+\frac{\hat{\theta}}{2}\right) W_x \cdot \left(t-\frac{\hat{\tau}}{2}, \omega-\frac{\hat{\theta}}{2}\right) e^{j\tau\hat{\theta}-j\theta\hat{\tau}}\, d\hat{\tau}\, d\hat{\theta}. \quad (1)$$

In [19], the marginals of the LAF were completely developed. Two of them are

$$2\pi |W_x(t, \omega)|^2 = \iint Q_x(t, \omega, \theta, \tau)\, d\tau\, d\theta$$
$$2\pi |A_x(\theta, \tau)|^2 = \iint Q_x(t, \omega, \theta, \tau)\, dt\, d\omega.$$

These marginal properties suggest that the LAF can be interpreted as a simultaneous distribution of the four variables corresponding to time, frequency, lag, and doppler. In [19], the geometry of the LAF was investigated for a two-component signal, and the LAF was applied to the design of signal adaptive kernels for the Cohen class. We will next present several new results concerning the LAF and apply it to the estimation of instantaneous chirp rate.

A. New Properties

The squared modulus of the ambiguity function is its own Fourier transform; this is referred to as Siebert's self-transform property [22], [23]

$$\frac{1}{2\pi} \iint |A_x(\theta, \tau)|^2 e^{j\hat{\tau}\theta - j\hat{\theta}\tau}\, d\tau\, d\theta = \left|A_x(\hat{\theta}, \hat{\tau})\right|^2.$$

It is straightforward to show that the LAF satisfies the same property for the τ and θ variables

$$\frac{1}{2\pi}\iint Q_x(t,\,\omega,\,\theta,\,\tau)\,e^{j\hat{\tau}\theta - j\hat{\theta}\tau}\,d\tau\,d\theta = Q_x(t,\,\omega,\,\hat{\theta},\,\hat{\tau}). \quad (2)$$

By combining (1) and (2), we arrive at a simpler formulation of the LAF

$$Q_x(t,\omega,\tau,\theta) = W_x\left(t+\frac{\tau}{2},\,\omega+\frac{\theta}{2}\right), W_x\left(t-\frac{\tau}{2},\,\omega-\frac{\theta}{2}\right). \quad (3)$$

With this knowledge, we can see that the LAF appears in Janssen's interference formula [24]

$$2\pi|W_x(t,\,\omega)|^2 = \iint W_x\left(t+\frac{\tau}{2},\,\omega+\frac{\theta}{2}\right) W_x\left(t-\frac{\tau}{2},\,\omega-\frac{\theta}{2}\right)\,d\tau\,d\theta.$$

Another peculiar, quartic function was defined by Szu and Caulfield [25]

$$Z_x(t,\,\omega,\,\tau,\,\theta) = \frac{1}{2\pi}\,W_x(t,\,\omega)\,A_x^*(\theta,\,\tau)\,e^{j\omega\tau - j\theta t}. \quad (4)$$

Szu and Caulfield were interested in using their function to compare the time-frequency content of two signals. By comparing (1) and (4), one can see that the relationship between Szu's function and the LAF is remarkably like the relationship between the Rihaczek distribution[2] and the Wigner distribution. The LAF also satisfies a relation similar to the Moyal formula [2], [3]

$$\iiiint Q_x(t,\,\omega,\,\tau,\,\theta)Q_y(t,\,\omega,\,\tau,\,\theta)\,dt\,d\omega\,d\tau\,d\theta = 4\pi^2\left|\int x(t)y^*(t)\,dt\right|^4.$$

B. Estimation of Instantaneous Chirp Rate

Let us suppose that we have a narrowband signal model of the form

$$x(t) = A(t)\,e^{j\varphi(t)} \quad (5)$$

where $A(t)$ is a slowly varying function, and the spectra of $A(t)$ and $e^{j\varphi(t)}$ do not overlap. The instantaneous frequency (IF) of these signals is often defined as[3]

$$\omega_i(t) = \dot{\varphi}(t)$$

and by extending this one step further, one can define the instantaneous chirp rate (ICR) as

$$c_i(t) = \ddot{\varphi}(t).$$

Several authors have proposed taking Radon (or Hough) transforms of the Wigner distribution [28]–[32] and the ambiguity function [33] as a means for estimating chirp rate. Here, we present examples that show that the method based on the Wigner distribution does not work well for multicomponent signals and for nonlinear chirps, whereas a similar method based on the LAF does work well.

We will use the following three functions:

$$\gamma_1(c,\,t_0,\,\omega_0) = \int W_x(t_0+r,\,\omega_0+rc)\,dr$$
$$\gamma_2(c,\,t_0,\,\omega_0) = \int |W_x(t_0+r,\,\omega_0+rc)|^2\,dr$$
$$\gamma_3(c,\,t_0,\,\omega_0) = \int Q_x(t_0,\,\omega_0,\,r,\,rc)\,dr$$

to define three estimators of ICR at the time–frequency point $(t_0,\,\omega_0)$

$$\hat{c}_i(t_0,\,\omega_0) = \arg\max_c \gamma_i(c,\,t_0,\,\omega_0) \qquad i=1,\,\cdots,\,3.$$

The quadratic estimator $\hat{c}_1$ has been proposed in several contexts [28]–[32]. If the signal is of the form $x(t) = e^{j(a_0+a_1t+a_2t^2)}$ (i.e., a linear chirp) and $(t_0,\,\omega_0)$ satisfy $\omega_0 = a_1 + a_2t_0$, then this estimator will be the maximum likelihood estimator [29]. The two quartic estimators $\hat{c}_2$ and $\hat{c}_3$ will be identical to each other when the signal is a noiseless, chirped Gaussian, but not for the general case.

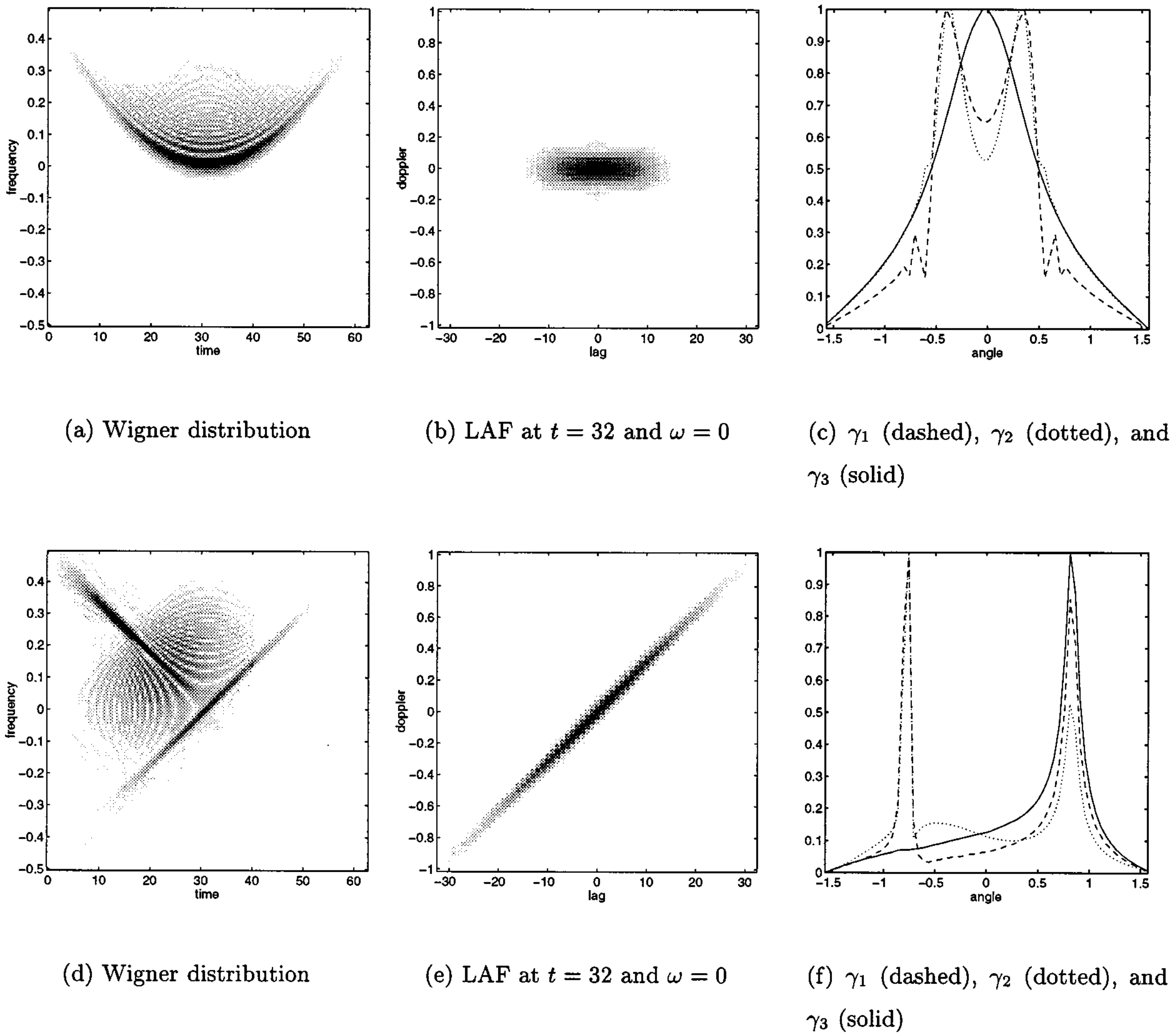
We now show two simple examples where the two estimators based on the Wigner distribution provide incorrect answers, whereas the estimator based on the LAF gives the correct answer. The first example is to estimate the ICR of a quadratic chirp. In Fig. 1(a), we show the Wigner distribution of a quadratic chirp, and we wish to estimate the ICR at the time–frequency point $(32,\,0)$. In Fig. 1(b), we show the LAF evaluated at the point $(32,\,0)$. Finally, in Fig. 1(c), we show the functions γ_1, γ_2, and γ_3. The true ICR at $(32,\,0)$ corresponds to an angle of 0 rad,[4] and only the LAF provides the correct estimate of this angle. For the second example, we repeat the same procedure in Fig. 1(d)–(f) for a signal composed of two chirps, where we would also like to estimate the ICR at the time–frequency point $(32,\,0)$. The chirp centered at $(32,\,0)$ corresponds to an angle of $\pi/4$ rad, and again, only the LAF provides the correct estimate of this angle. The other two estimators give an estimate corresponding to the other chirp centered at $(16,\,0.25)$.

The LAF-based estimator correctly determined the ICR for the noiseless, quadratic chirp in the above example. By a geometrical argument, we expect this to always approximately be true. In Fig. 2, we show an example that illustrates the computation of the LAF at a given time–frequency point, where the point of interest is indicated by the circle. By analyzing (3), one can see that the LAF at a given time-frequency point is simply the product of the Wigner distribution with its reflection about this point. Intuitively, this multiplication of the Wigner distribution with its reflection serves to eliminate the curvature and emphasize the tangential information at this point.

[2]Note that the Wigner distribution is the two-dimensional (2-D) Fourier transform of the ambiguity function and the definition of the Rihaczek distribution $R_x(t,\,\omega) = x(t)\,X^*(\omega)\,e^{-j\omega t}$

[3]There is much research and much debate about how to define the analytic signal and instantaneous frequency which is beyond the scope of this paper [1], [2], [26], [27].

[4]The actual values of the chirp rates are not very intuitive so we plot the functions γ_i as a function of the angle in the time–frequency plane, where the angle is measured counterclockwise from the positive time axis.



(a) Wigner distribution

(b) LAF at $t = 32$ and $\omega = 0$

(c) γ_1 (dashed), γ_2 (dotted), and γ_3 (solid)

(d) Wigner distribution

(e) LAF at $t = 32$ and $\omega = 0$

(f) γ_1 (dashed), γ_2 (dotted), and γ_3 (solid)

Fig. 1. Comparison of three estimators of instantaneous chirp rate.

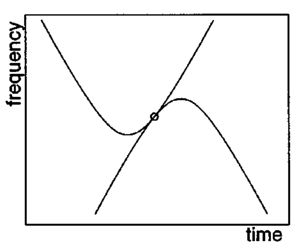


Fig. 2. Geometrical argument for the LAF based estimator.

For the LAF-based estimator to correctly determine the ICR, it must satisfy the following condition:

$$\frac{d}{dc}\gamma_3(c,\, t,\, \dot{\varphi}(t))\bigg|_{c=\ddot{\varphi}(t)} = 0. \tag{6}$$

This condition states that the function γ_3 will have a local maximum or minimum at the true value of the chirp rate. In the Appendix, we show that for quadratic chirps, the LAF-based estimator always satisfies the above necessary condition. We have not been able to show that (6) corresponds to a global maximum, but given the above geometrical argument, we expect this to be true.

To compare the statistical performance of the three estimators, we ran simulations consisting of a chirped Gaussian in complex, white, Gaussian noise (CWGN). The discrete signal used in the simulation is

$$\left(\sqrt{2\pi}\, d\right)^{-(1/2)} \cdot \exp\left\{-\left(\frac{n-t}{2\,d}\right)^2 + j\frac{c}{2}(n-t)^2 + j\omega(n-t)\right\}$$

with $n = [1, \cdots, 64]$, $t = 32$, $\omega = 0$, $c = 2\pi/64$, and $d = 8$. In practice, the location of the chirp will generally not be known and will have to be estimated. Thus, we incorporated into the simulations a parameter Δ that indicates the error in estimating the location of the chirp in time. In Table I, we present the mean and variance of the three estimators for 1000 trials (with one exception that will be explained below), where the column σ corresponds to the standard deviation of the CWGN. In several instances, the LAF-based estimator $\hat{c}_3$ outperforms $\hat{c}_1$, which has a clear relationship with the MLE [29]. The estimator $\hat{c}_2$ does not perform as well as the other two; therefore, it will not be examined further.

TABLE I
BIAS AND VARIANCE OF THE THREE LOCAL ESTIMATORS OF CHIRP RATE

		100x bias			10000x variance		
Δ	σ	W	W^2	Q	W	W^2	Q
0	0.02	0.00	0.00	0.00	0.00	0.00	0.00
0	0.05	0.00	0.00	0.00	0.00	0.00	0.00
0	0.10	0.04	0.02	0.02	0.11	0.07	0.07
0	0.15	0.20	0.46	0.18	1.21	3.64	1.64
$\frac{1}{2}$	0.02	0.00	0.00	0.00	0.00	0.04	0.00
$\frac{1}{2}$	0.05	0.05	0.17	0.01	0.05	0.21	0.03
$\frac{1}{2}$	0.10	0.10	0.19	0.06	0.36	0.62	0.41
1	0.02	0.91	1.66	0.12	0.03	0.09	0.42
1	0.05	0.70	1.21	0.24	0.67	0.86	2.89
1	0.10	0.81	0.96	1.07	2.94	2.28	12.2

- For $\Delta = 0$ and $\sigma = 0.10$, the LAF-based estimator has a lower bias and variance than $\hat{c}_1$. The number of trials for this example was increased to 5000 to obtain statistical significance.[5]
- For $\Delta = (1/2)$ and $\sigma = 0.05$, the LAF-based estimator has a lower bias and variance than $\hat{c}_1$. These differences are statistically significant with 1000 trails.
- For $\Delta = 1$ and $\sigma =$0.02 or 0.05, the LAF-based estimator has a much lower bias than $\hat{c}_1$. These differences are statistically significant with 1000 trails.
- With very low SNR, the LAF-based estimator will have a higher variance than $\hat{c}_1$.

The LAF-based estimator was used in [34] for the problem of atomic decomposition with chirped Gabor functions.

III. QUARTIC, SHIFT-COVARIANT DISTRIBUTIONS

In this section, we derive two types of quartic, shift-covariant distributions. The first is a distribution of time and frequency and is derived by imposing shift covariance properties [2]. The second is a distribution of time and chirp-rate and is derived using a property of cubic polynomials.

A. Time and Frequency

The class of quadratic, shift-covariant TFDs can be expressed in several forms[6]

$$\begin{aligned} C_x(t, \omega;\ \phi) &= \iint x(t_1)\, x^*(t_2)\, \psi(t_1 - t,\, t_2 - t) \\ &\quad \cdot e^{-j\omega(t_1 - t_2)}\, dt_1\, dt_2 \\ &= \iint W_x(s,\, v)\, \phi(s - t,\, v - \omega)\, ds\, dv \\ &= \frac{1}{2\pi} \iint A_x(\theta,\, \tau)\, \Phi(\theta,\, \tau)\, e^{j\tau\omega - j\theta t}\, d\theta\, d\tau \end{aligned} \tag{7}$$

and is commonly referred to as the Cohen class [35]. The above can be derived axiomatically [2], and we will apply the same procedure here. The class of general, *quartic* TFDs can be expressed in the following form:[7]

$$\begin{aligned} T_x(t, \omega;\ K) = \iiiint & K(t,\, \omega;\ t_1,\, t_2,\, t_3,\, t_4) \\ & \cdot x(t_1)\, x(t_2)\, x^*(t_3)\, x^*(t_4)\, dt_1\, dt_2\, dt_3\, dt_4. \end{aligned}$$

If we impose time and frequency shift covariance, i.e., $y(t) = x(t - t_0)\, e^{j\omega_0 t}$ constrains the corresponding TFDs as $T_y(t, \omega;\ K) = T_x(t - t_0,\, \omega - \omega_0;\ K)$, then we arrive at the quartic, shift-covariant class

$$\begin{aligned} F_x(t, \omega;\ \psi) = \iiiint & x(t_1)\, x(t_2)\, x^*(t_3)\, x^*(t_4) \psi \\ & \cdot (t_1 - t,\, t_2 - t,\, t_3 - t,\, t_4 - t) \\ & \cdot e^{-j\omega(t_1 + t_2 - t_3 - t_4)}\, dt_1\, dt_2\, dt_3\, dt_4 \end{aligned} \tag{8}$$

which we will denote as the quartic class of TFDs (the shift covariance will be implied). The quartic class has a dual form, which is expressed in terms of the Fourier transform of the signal and can also be expressed in terms of the LAF discussed in the previous section

$$\begin{aligned} F_x(t, \omega;\ \phi) = \iiiint & Q_x(s,\, \nu,\, \tau,\, \theta) \phi \\ & \cdot (s - t,\, \nu - \omega,\, \tau,\, \theta)\, ds\, d\nu\, d\tau\, d\theta. \end{aligned} \tag{9}$$

Thus, the LAF is a generating function for the quartic class in the same way that the Wigner distribution is a generating function for the Cohen class. Note that Szu's function is also a generating function for the quartic class, as is the Rihaczek distribution for the Cohen class.

Another form that will prove to be useful in subsequent sections is based on what we will call the *ambiguous ambiguity function* (AAF)

$$\mathcal{Q}_x(\hat{\tau},\, \hat{\theta},\, \tau,\, \theta) = \frac{1}{2\pi} \iint Q_x(t,\, \omega,\, \tau,\, \theta)\, e^{j\omega\hat{\tau} - jt\hat{\theta}}\, dt\, d\omega.$$

The AAF has a simple relationship with the ambiguity function

$$\mathcal{Q}_x(\hat{\tau},\, \hat{\theta},\, \tau,\, \theta) = A_x\left(\theta + \frac{\hat{\theta}}{2},\, \tau + \frac{\hat{\tau}}{2}\right) A_x^*\left(\theta - \frac{\hat{\theta}}{2},\, \tau - \frac{\hat{\tau}}{2}\right).$$

The relationship between the LAF and the AAF is similar to that between the Wigner distribution and the ambiguity function.[8] With the following transformation on the kernel:

$$\Phi(\hat{\tau},\, \hat{\theta},\, \tau,\, \theta) = \frac{1}{2\pi} \iint \phi(t,\, \omega,\, \tau,\, \theta)\, e^{j\omega\hat{\tau} - jt\hat{\theta}}\, dt\, d\omega$$

we can formulate the quartic class as

$$\begin{aligned} F_x(t, \omega;\ \Phi) = \frac{1}{2\pi} \iiiint & \mathcal{Q}_x(\hat{\tau},\, \hat{\theta},\, \tau,\, \theta) \Phi(\hat{\tau},\, \hat{\theta},\, \tau,\, \theta) \\ & \cdot e^{jt\hat{\theta} - j\omega\hat{\tau}}\, d\hat{\tau}\, d\hat{\theta}\, d\tau\, d\theta. \end{aligned} \tag{10}$$

[5]We assumed the estimators are Gaussian distributed and used the estimated means and variances to compute 95% confidence intervals for the means and variances. If the 95% confidence intervals do not overlap, then we conclude that the differences are statistically significant.

[6]We will use the notations $C_x(t,\, \omega;\ \psi)$, $C_x(t,\, \omega;\ \phi)$, and $C_x(t,\, \omega;\ \Phi)$ interchangeably to emphasize one form over the other.

[7]There is ambiguity in deciding how many of the signal terms should be conjugated. Other choices result in distributions with less interesting properties.

[8]Recall the self-transforming property of the LAF.

The three forms of the quartic class in (8)–(10) are analogous, respectively, to the three forms for the Cohen class in (7).

1) Examples: Up to a multiplicative constant, every Cohen class TFD is also in the quartic class. If we denote the signal energy as $E_x = \int |x(t)|^2\,dt$, then $F_x(t, \omega;\ \Phi_1) = E_x\, C_x(t, \omega;\ \Phi_2)$ with the kernels being related as

$$\Phi_1(\hat{\tau}, \hat{\theta}, \tau, \theta) = \delta\left(\tau - \frac{\hat{\tau}}{2}\right)\delta\left(\theta - \frac{\hat{\theta}}{2}\right)\,\Phi_2(\hat{\theta}, \hat{\tau}).$$

The squared modulus of any Cohen class TFD will be a member of the quartic class.

$$F_x(t, \omega;\ \phi_1) = |C_x(t, \omega;\ \phi_2)|^2$$

where the kernels are related as

$$\phi_1(t, \omega, \tau, \theta) = \phi_2\left(t + \frac{\tau}{2}, \omega + \frac{\theta}{2}\right)\,\phi_2^*\left(t - \frac{\tau}{2}, \omega - \frac{\theta}{2}\right).$$

An interesting example is the squared modulus of the Rihaczek distribution, which will be positive and satisfy marginal properties. There is no Cohen class TFD that satisfies these two properties [2].[9]

The product of any two Cohen class TFDs will be in the quartic class.

$$F_x(t, \omega;\ \phi_1) = C_x(t, \omega;\ \phi_2)\, C_x(t, \omega;\ \phi_3)$$

where the kernels are related as

$$\phi_1(t, \omega, \tau, \theta) = \phi_2\left(t + \frac{\tau}{2}, \omega + \frac{\theta}{2}\right)\,\phi_3\left(t - \frac{\tau}{2}, \omega - \frac{\theta}{2}\right).$$

An interesting example concerns the strong time support and the strong frequency support properties [36]. The only Cohen class TFDs to satisfy both of these are the linear combinations of the real and imaginary parts of the Rihaczek distribution [37], [38]. However, if $C_x(t, \omega;\ \phi_2)$ satisfies the strong time support property, and $C_x(t, \omega;\ \phi_3)$ satisfies the strong frequency support property, then their product will be in the quartic class and satisfy both properties.

The following convolutions of two Cohen class TFDs will be in the quartic class

$$\begin{aligned} F_x(t, \omega;\ \phi_1) &= C_x(2t, 2\omega;\ \phi_2) *_t *_\omega C_x(2t, 2\omega;\ \phi_3)\\ F_x(t, \omega;\ \phi_1') &= C_x(2t, \omega;\ \phi_2) *_t\ C_x(2t, \omega;\ \phi_3)\\ F_x(t, \omega;\ \phi_1'') &= C_x(t, 2\omega;\ \phi_2) *_\omega\ C_x(t, 2\omega;\ \phi_3). \end{aligned}$$

The second-order L-Wigner distribution [11], [12] and its dual form are members of the quartic class

$$\begin{aligned} L_x(t, \omega) &= \int x\left(t + \frac{\tau}{4}\right)^2 x^*\left(t - \frac{\tau}{4}\right)^2 e^{-j\omega\tau}\,d\tau\\ \hat{L}_x(t, \omega) &= \frac{1}{2\pi}\int X\left(\omega + \frac{\theta}{4}\right)^2 X^*\left(\omega - \frac{\theta}{4}\right)^2 e^{j\theta t}\,d\theta \end{aligned}$$

with kernels corresponding, respectively, to

$$\begin{aligned} \phi(t, \omega, \tau, \theta) &= \delta(t)\,\delta(\omega)\,\delta(\theta)\\ \phi(t, \omega, \tau, \theta) &= \delta(t)\,\delta(\omega)\,\delta(\tau). \end{aligned}$$

The intersection of the polynomial Wigner–Ville distributions (PWVD's) [13], [14] and the quartic class is

$$P_x(t, \omega) = \int R(t, \tau)\, e^{-j\omega\tau}\,d\tau \tag{11a}$$

$$R_x(t, \tau) = x(t + b_0\tau)\, x(t + b_1\tau)\, x^*(t + b_2\tau)\, x^*(t + b_3\tau) \tag{11b}$$

and will be discussed in more detail below.

Several authors have defined TFDs based on higher order spectra [9], [10], [39], [40] In particular, the fourth-order Cohen class (4-CC) [9], [10] was derived by imposing covariance properties and has the closest connection to the quartic class defined here. The 4-CC has three frequency variables, whereas the quartic class has only one frequency variable, and there are many ways to reduce these three variables to one. For example, if $H_x(t, \omega_1, \omega_2, \omega_3)$ is a member of 4-CC, then

$$\begin{aligned} F_x(t, \omega;\ \phi_1) &= H_x(t, \omega, \omega, \omega)\\ F_x(t, \omega;\ \phi_2) &= \iint H_x(t, \omega, \omega_2, \omega_3)\,d\omega_2\,d\omega_3 \end{aligned}$$

are members of the quartic class. Since there are many ways of reducing 4-CC to the quartic class, there is no simple way to prescribe a precise relationship between the two.

2) Localized Distributions: We are now going to investigate the intersection of the quartic class and the PWVD's, which are expressed in (11), when the signal is of the form

$$x(t) = e^{j\varphi(t)} \tag{12}$$

and $\varphi(t)$ is an arbitrary cubic polynomial [i.e., $x(t)$ is a quadratic chirp]. We are interested in finding a perfectly localized distribution of the form

$$P_x(t, \omega) = 2\pi\,\delta(\omega - \dot{\varphi}(t)).$$

There does not exist a Cohen class TFD that localizes this class of signals [2].[10]

When $\varphi(t)$ is a cubic polynomial, (11b) simplifies to

$$\begin{aligned} j\,\log R_x(t, \tau) &= \tau\,\dot{\varphi}(t)\,(b_0 + b_1 - b_2 - b_3)\\ &\quad + \tfrac{1}{2}\,\tau^2\,\ddot{\varphi}(t)\,(b_0^2 + b_1^2 - b_2^2 - b_3^2)\\ &\quad + \tfrac{1}{6}\,\tau^3\,\dddot{\varphi}(t)\,(b_0^3 + b_1^3 - b_2^3 - b_3^3). \end{aligned} \tag{13}$$

Clearly, if we can solve the following system of equations:

$$\begin{aligned} b_0 + b_1 - b_2 - b_3 &= 1\\ b_0^2 + b_1^2 - b_2^2 - b_3^2 &= 0\\ b_0^3 + b_1^3 - b_2^3 - b_3^3 &= 0 \end{aligned}$$

then $R_x(t, \tau) = e^{j\tau\dot{\varphi}(t)}$, $P_x(t, \omega) = 2\pi\,\delta(\omega - \dot{\varphi}(t))$, and we will have the ideal distribution. The solution to the above system is

$$\begin{aligned} b_1 &= \frac{1}{3}\,\frac{3b_0 - 1}{2b_0 - 1}\\ b_2 &= \frac{-1 + b_0 + b1 \pm \sqrt{(b_0 - b_1)^2 + 2b_0 + 2b_1 - 1}}{2}\\ b_3 &= b_0 + b_1 - b_2 - 1 \end{aligned}$$

[9]With the more general definition of the Cohen class [1], there exist TFDs that satisfy these two properties [1], [17].

[10]Localized distributions can be obtained with the more general definition of the Cohen class that includes signal-dependent kernels [1].

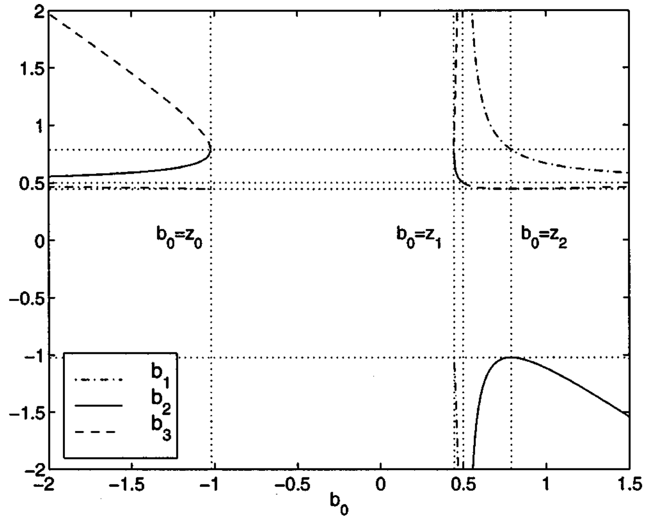


Fig. 3. Parameters that localize quadratic chirps.

which leaves b_0 as a free parameter. If

$$b_0 \geq \frac{\sqrt{3}}{6}\left(-1+\sqrt{3+2\sqrt{3}}\right) \tag{14a}$$

or

$$b_0 \leq \frac{\sqrt{3}}{6}\left(-1+\sqrt{3+2\sqrt{3}}\right) \tag{14b}$$

then b_1, b_2, and b_3 will be real. The entire set of parameters is displayed in Fig. 3 as a function of b_0.

While all values of b_0 [subject to (14)] will furnish the ideal distribution for quadratic chirps, the following three values for b_0:

$$z_0 = \frac{\sqrt{3}}{6}\left(-1-\sqrt{3+2\sqrt{3}}\right) \tag{15a}$$

$$z_1 = \frac{\sqrt{3}}{6}\left(-1+\sqrt{3+2\sqrt{3}}\right) \tag{15b}$$

$$z_2 = \frac{1}{2}+\frac{\sqrt{3}}{6} \tag{15c}$$

give the same distribution with three additional, interesting properties.

First, two of the four coefficients will be the same, which simplifies the computations.

Second, the distribution of a Gaussian signal $y(t)=e^{-t^2}$ will have the highest concentration in the time–frequency plane. The distribution of this Gaussian is

$$P_y(t,\omega) = e^{-4f(b_0)t^2}\, e^{-g(b_0)\omega^2/4}\, e^{jt\omega k(b_0)}$$

$$f(b_0) = 1-\frac{1}{8}\,\frac{144b_0^4-144b_0^3+60b_0^2-12b_0+1}{36b_0^4-36b_0^3+18b_0^2-6b_0+1}$$

$$g(b_0) = 18\frac{(-1+2b_0)^2}{36b_0^4-36b_0^3+18b_0^2-6b_0+1}$$

where $f(\,)$, $g(\,)$, and $k(\,)$ are real functions of b_0. To minimize the concentration, we want to find the value of b_0 that maximizes the functions $f(\,)$ and $g(\,)$ [$k(\,)$ is irrelevant since it does not affect the concentration]. Subject to (14), each of these functions has equal maxima at z_0, z_1, and z_2.

Third, the distributions seem to be more localized for general signals. By performing a Taylor series expansion

$$\varphi(t+b\tau) = \sum_{i=0}^{\infty} (b\tau)^i\, \varphi^{(i)}(t)/i!$$

we can express (13) in a more general form

$$j\,\log R_x(t,\tau) = \sum_{i=1}^{\infty} \tau^i\, \varphi^{(i)}(t)\, h_i(b_0)/i!.$$

From the above. we know that $h_1(b_0)=1$, $h_2(b_0)=0$, and $h_3(b_0)=0$. If there existed a value $\hat{b}$ such that $h_i(\hat{b})=0$ for $3<i<p$, then we would have the ideal distribution when $\varphi(t)$ is a polynomial of degree p. Unfortunately, the function $h_4(b_0)$ is always positive, and therefore, we will not be able to provide localization when $\varphi(t)$ has a degree higher than 3. However, it can be shown that both $h_4(b_0)$ and $h_5(b_0)$ have equal minima at z_0, z_1, and z_2. Thus, the distribution will be, in a sense, "optimized" when $\varphi(t)$ is a fourth- or fifth-order polynomial. We have not shown any general results on the minimization of the $h_i(b_0)$ for $i>5$.

In Fig. 4, examples of the above distribution, with $b=z_0$, are shown for a quadratic chirp, a cubic chirp, and a sinusoid. The distribution of the quadratic chirp is perfectly localized, whereas the distribution of the cubic chirp is not. The sinusoid has many cross terms that appear from the interaction of the positive and negative frequency components. Since some of the cross terms are not oscillatory, simple filtering techniques will not attenuate them, and more sophisticated measures are necessary. The analysis of the geometry of the LAF in [19] suggests that to eliminate cross terms, the kernel $\phi(t,\omega,\tau,\theta)$ should be a lowpass filter in the t and ω variables and a window with compact support in the τ and θ variables. Another method for attenuating cross terms in polynomial Wigner distributions has been discussed in [41].

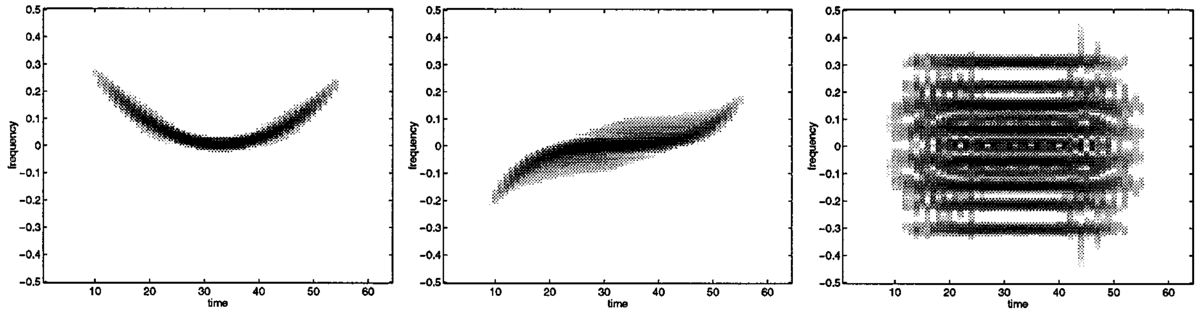
In Fig. 5, we show three different distributions of a signal with a sinusoidal instantaneous frequency. The first is the Wigner distribution, the second is the localized distribution with $b_0=z_0$, and the third is the localized distribution with $b_0=1$. The localized distribution appears more localized for $b_0=z_0$ than for $b_0=1$.

3) Properties: There are an abundance of properties that have been applied to TFDs. To exhaustively show these properties for the quartic case is difficult and time and space consuming, and thus, we show a few of the more interesting ones here.

a) Marginals: There are several marginals that one could consider for quartic TFDs. In Table II, we list four quartic TFDs and their corresponding marginals. All of the marginals contain cross terms (like those for the Cohen class), but the cross terms will be much more prominent for the first, third, and fourth cases, and thus, the second will likely be preferred in most instances.

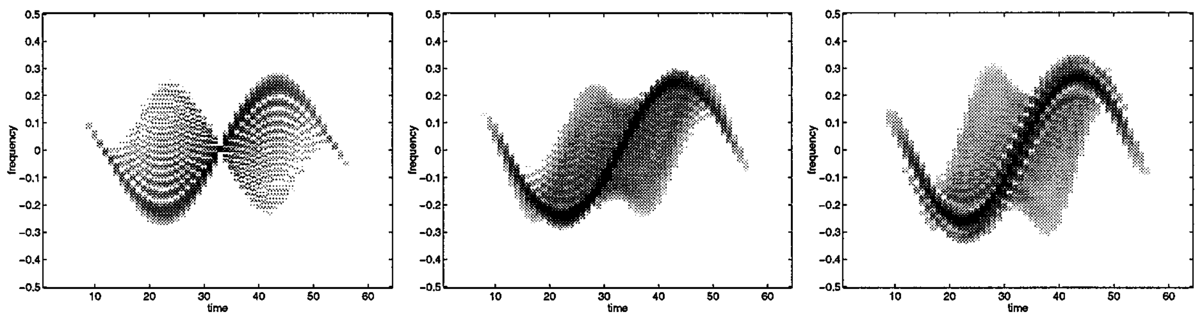
$$\Phi(0,\hat{\theta},\tau,\theta) = \delta\left(\theta-\frac{\hat{\theta}}{2}\right)\,\delta(\tau)$$

$$\Phi(\hat{\tau},0,\tau,\theta) = \delta\left(\tau-\frac{\hat{\tau}}{2}\right)\,\delta(\theta)$$



(a) quadratic chirp (b) cubic chirp (c) sinusoid

Fig. 4. Examples of the localized distributions.



(a) Wigner Distribution (b) Localized Distribution – $b_0 = z_0$ (c) Localized Distribution – $b_0 = 1$

Fig. 5. Comparison of the Wigner distribution and the localized distributions.

are, respectively, sufficient kernel constraints for the following marginal properties:

$$\int T_x(t,\,\omega;\;\Phi)\,d\omega = 2\pi\,E_x\,|x(t)|^2$$
$$\int T_x(t,\,\omega;\;\Phi)\,dt = E_x\,|X(\omega)|^2$$

b) Moyal formula: There do not exist any TFDs in the quartic class that satisfy the following Moyal-like formula

$$\iint F_x(t,\,\omega;\;\Phi)\,F_y^*(t,\,\omega;\;\Phi)\,dt\,d\omega = 4\pi^2\left|\int x(t)y^*(t)\,dt\right|^4.$$

If we assume the above holds for some Φ, then we have

$$\iiint\iiint \mathcal{Q}_x(\tau,\,\theta,\,\hat{\tau},\,\hat{\theta})\,\mathcal{Q}_y^*(\tau,\,\theta,\,\tilde{\tau},\,\tilde{\theta})\,\Phi(\tau,\,\theta,\,\hat{\tau},\,\hat{\theta})\Phi^* \cdot(\tau,\,\theta,\,\tilde{\tau},\,\tilde{\theta})\,d\tau\,d\theta\,d\hat{\tau}\,d\hat{\theta}\,d\tilde{\tau}\,d\tilde{\theta}$$
$$= \iiiint \mathcal{Q}_x(\tau,\,\theta,\,\hat{\tau},\,\hat{\theta})\,\mathcal{Q}_y^*(\tau,\,\theta,\,\hat{\tau},\,\hat{\theta})\,d\tau\,d\theta\,d\hat{\tau}\,d\hat{\theta}$$

which implies that

$$\Phi(\tau,\,\theta,\,\hat{\tau},\,\hat{\theta})\Phi^*(\tau,\,\theta,\,\tilde{\tau},\,\tilde{\theta}) = \delta(\hat{\tau}-\tilde{\tau})\,\delta(\hat{\theta}-\tilde{\theta})$$

which can not be true.

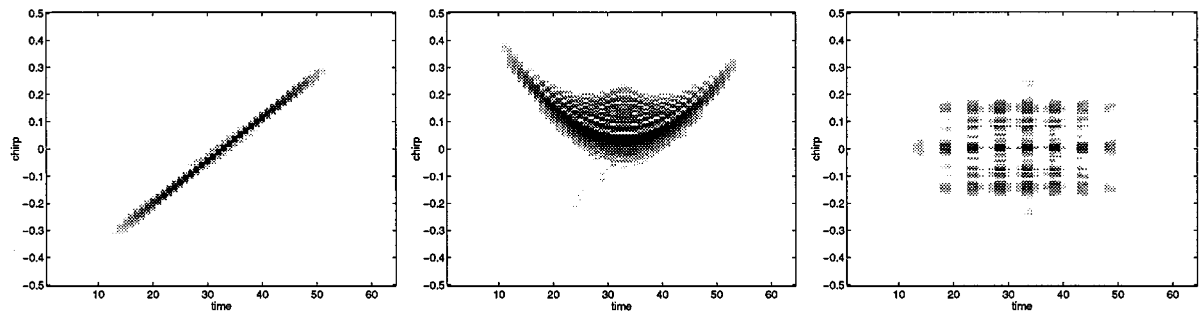
c) Symplectic transformations: The Wigner distribution is the only quadratic distribution covariant to symplectic transformations [42]. For the quartic class, the following are, respectively, sufficient kernel constraints for TFDs in the quartic class to be covariant to scalings, Fourier transforms, and chirp multiplications

$$\Phi(\hat{\tau},\,\hat{\theta},\,\tau,\,\theta) = \Phi(a\hat{\tau},\,\hat{\theta}/a,\,a\tau,\,\theta/a)$$
$$\Phi(\hat{\tau},\,\hat{\theta},\,\tau,\,\theta) = \Phi(-\hat{\theta},\,\hat{\tau},\,-\theta,\,\tau)$$
$$\Phi(\hat{\tau},\,\hat{\theta},\,\tau,\,\theta) = \Phi(\hat{\tau},\,\hat{\theta}-a\hat{\tau},\,\tau,\,\theta-a\tau).$$

If $\Phi(\hat{\tau},\,\hat{\theta},\,\tau,\,\theta) = f(\hat{\theta}\tau - \theta\hat{\tau})$ for some function $f(\,)$, then the distribution will be covariant to the above three transformations and, thus, all symplectic transformations. Two distributions from the quartic class that satisfy this are $E_x\,W_x(t,\,\omega)$ and $W_x(t,\,\omega)^2$.

TABLE II
MARGINALS OF SEVERAL QUARTIC TFDs

Distribution	Time Marginal	Frequency Marginal
$W_x^2(t,\omega)$	$4\pi\,\lvert x(2t)\rvert^2 * \lvert x(2t)\rvert^2$	$\frac{1}{\pi}\,\lvert X(2\omega)\rvert^2 * \lvert X(2\omega)\rvert^2$
$E_x\,W_x(t,\omega)$	$2\pi E_x\,\lvert x(t)\rvert^2$	$E_x\,\lvert X(\omega)\rvert^2$
$L_x(t,\omega)$	$2\pi\,\lvert x(t)\rvert^4$	$2\left\lvert\frac{1}{2\pi}X(2\omega) * X(2\omega)\right\rvert^2$
$\tilde{L}_x(t,\omega)$	$4\pi\,\lvert x(2t) * x(2t)\rvert^2$	$\lvert X(\omega)\rvert^4$



(a) quadratic chirp (b) cubic chirp (c) sinusoid

Fig. 6. Examples of the time-chirp distribution.

B. Time and Chirp Rate

Consider the time-shift and chirp-shift operators

$$(\boldsymbol{T}_{t_o} x)(t) = x(t - t_o)$$
$$(\boldsymbol{C}_{c_o} x)(t) = x(t)\, e^{j c_o t^2/2}.$$

According to the results of Baraniuk [43], there do not exist quadratic distributions covariant to these two operators since these two operators are unitarily equivalent to neither time-shift and frequency-shift nor to time-shift and scale. However, we will derive a quartic function covariant to these two operators.

Note that if $\varphi(t)$ is a quadratic polynomial, then

$$\varphi\left(t + \frac{\tau}{2}\right) - \varphi\left(t - \frac{\tau}{2}\right) = \tau\, \dot{\varphi}(t)$$

holds exactly for all t and τ. From this, it is clear that the Wigner distribution provides the ideal distribution for linear chirps

$$W_x(t, \omega) = \int x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-j\omega\tau}\, d\tau = \int e^{-j\tau(\omega - \dot{\varphi}(t))}\, d\tau = 2\pi\, \delta(\omega - \dot{\varphi}(t)).$$

If $\varphi(t)$ is a cubic polynomial, then

$$\varphi(t + \tau) - 2\varphi(t) + \varphi(t - \tau) = \tau^2\, \ddot{\varphi}(t)$$

holds exactly for all t and τ. From the above, one can derive the following distribution of time and chirp rate:

$$Z_x(t, \tau) = \begin{cases} x\left(t + \sqrt{\tau}\right) x^*(t) x^*(t) x\left(t - \sqrt{\tau}\right), & \tau \geq 0 \\ Z_x^*(t, -\tau), & \tau < 0 \end{cases}$$
$$T_x(t, c) = \int Z_x(t, \tau)\, e^{-jc\tau}\, d\tau$$

that will provide the ideal time-chirp distribution for quadratic chirps [as defined in (12) with $\varphi(t)$ a cubic polynomial]

$$T_x(t, c) = \int e^{-j\tau(c - \ddot{\varphi}(t))}\, d\tau = 2\pi\, \delta(c - \ddot{\varphi}(t)).$$

This distribution is covariant to the time-shift and chirp-shift operators. If $y(t) = (\boldsymbol{T}_{t_o} \boldsymbol{C}_{c_o} x)(t)$, then $T_y(t, c) = T_x(t - t_o, c - c_o)$. The distribution will also be invariant to frequency shifts.

In Fig. 6, examples of the time-chirp distribution are shown for the same three signals as in Fig. 4. As a result, the instantaneous chirp rate in the three time-chirp distributions are roughly the derivative of the instantaneous frequency of the three localized distributions. For the quadratic chirp, the distribution is localized along the linear, instantaneous chirp rate. For the cubic chirp, the instantaneous chirp rate is a quadratic function, although the distribution is not perfectly localized. The chirp rate of the sinusoid is zero, and thus, the two auto terms appear at $c = 0$ on the chirp axis. The cross terms have a complicated structure and obscure the auto terms.

C. Discussion

The quartic distributions introduced here have properties that are not obtainable with quadratic distributions. However, the quartic nature of these distributions presents other complications.

The general quartic class of TFDs will have many more cross terms than quadratic distributions. The analysis of the cross terms in the LAF in [19] suggests a lowpass filtering kernel in time and frequency and a windowing kernel in lag and doppler. However, the manipulation of these four-dimensional functions will likely be prohibitive computationally. The localized distributions provide one means of reducing the dimensionality and the computational complexity, and perhaps other methods of reducing the dimensionality will also lead to interesting distributions.

The localized distributions have the same computational complexity as quadratic distributions in the Cohen class. While these distributions provide interesting representations for narrowband signals of the form (5), there are many more cross terms for multicomponent signals, some of which are not oscillatory. A method for attenuating cross terms in the PWVD's (of which the localized distributions are a subset) has been proposed in [41]. A numerical difficulty is that the localized distributions require interpolation of the signal to irrational factors.

The time-chirp distribution has the same computational complexity as distributions in the Cohen class and provides interesting distributions for narrowband signals, but it has many cross terms for multicomponent signals and requires interpolation to irrational factors. The cross terms in the time-chirp distribution have a complicated structure and an analysis of the interferences [44], [45] could lead to methods for their attenuation.

IV. CONCLUSIONS

In this paper, we have extolled the virtues of several types of quartic TFDs. First, we presented new results on the local ambiguity function and showed that it can provide a better estimate of instantaneous chirp rate than the Wigner distribution, particularly in high SNR and for nonlinear chirps. Second, we introduced the class of quartic, shift-covariant TFDs (the quartic class) and showed that its members can have properties that are not obtainable with the Cohen class. An example that was presented in detail is the localization of quadratic chirps. Third, we presented a covariant distribution of time and chirp-rate, which is not possible with quadratic distributions.

The quartic distributions also present complicated vices: those of increased computational complexity and a greater number of cross terms. In this paper, we expose these vices without providing any significant contributions for overcoming them. However, we provide a framework for further investigation of the quartic distributions in the same way that the Cohen class has provided the means for creating quadratic distributions that ameliorate the vices of the Wigner distribution. Other authors have recently presented methods for ameliorating the vices of quartic distributions [14], [41], and it is hoped that our results will engender new research in this area.

MATLAB software for implementing several of these quartic distributions is available at http://mdsp.bu.edu/jeffo.

APPENDIX

Our signal is a quadratic chirp, and thus, $x(t) = e^{j\varphi(t)}$, and $\varphi(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. The Wigner distribution of this signal is

$$W_x(t,\,\omega) = \int e^{j(a_3\tau^3/4 - \tau(\omega - \dot{\varphi}(t)))}\,d\tau.$$

We show here that for the above signal and any time–frequency point along the IF of the signal $(t,\,\dot{\varphi}(t))$, the LAF-based estimator satisfies the following necessary condition:

$$\frac{d}{dc}\,\gamma_3(c,\,t,\,\dot{\varphi}(t))\bigg|_{c=\ddot{\varphi}(t)} = 0.$$

We start with

$$\begin{aligned}\gamma_3(c,\,t,\,\dot{\varphi}(t)) &= \int Q_x(t,\,\dot{\varphi}(t),\,r,\,rc)\,dr \\ &= \iiint \exp\{j(a_3(u^3 - v^3)/4 \\ &\qquad - (u+v)r(c - \ddot{\varphi}(t))/2 \\ &\qquad + (u-v)3a_3 r^2/4)\}\,du\,dv\,dr \\ &= \iint \sqrt{\frac{j\pi}{(u-v)3a_3/4}} \\ &\quad \cdot \exp\left\{j\left(a_3(u^3 - v^3)\Big/4 - \frac{(u+v)^2(c-\ddot{\varphi}(t))^2}{12(u-v)a_3}\right)\right\} dudv\end{aligned}$$

where the last step comes from the Fourier transform of a chirp signal. Applying the derivative results in

$$\begin{aligned}\frac{d}{dc}\gamma_3(c,t,\dot{\varphi}(t)) &= \iint \sqrt{\frac{j\pi}{(u-v)3a_3/4}}\,\frac{(u+v)^2(c-\ddot{\varphi}(t))}{6(u-v)a_3} \\ &\quad \cdot \exp\left\{j\left(a_3(u^3-v^3)\Big/4 - \frac{(u+v)^2(c-\ddot{\varphi}(t))^2}{12(u-v)a_3}\right)\right\} dudv\end{aligned}$$

which will be 0 when $c = \ddot{\varphi}(t)$.

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REFERENCES

[1] L. Cohen, *Time-Frequency Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[2] P. Flandrin, *Time-Frequency/Time-Scale Analysis*. San Diego, CA: Academic, 1999.

[3] F. Hlawatsch and G. F. Boudreaux-Bartels, "Linear and quadratic time-frequency signal representations," *IEEE Signal Processing Mag.*, pp. 21–67, Apr. 1992.

[4] M. B. Priestley, *Non-Linear and Non-Stationary Time-Series Analysis*. New York: Academic, 1988.

[5] "Time-frequency analysis: Biomedical, acoustical, and industrial applications," *Proc. IEEE*, vol. 84, Sept. 1996.

[6] S. G. Mallat, *A Wavelet Tour of Signal Processing*. New York: Academic, 1998.

[7] A. Papandreou, F. Hlawatsch, and G. F. Boudreaux-Bartels, "The hyperbolic class of QTFRs—Part I: Constant-Q warping, the hyperbolic paradigm, properties, and members," *IEEE Trans. Signal Processing*, vol. 41, pp. 3425–3444, Dec. 1993.

[8] J. Bertrand and P. Bertrand, "A class of affine Wigner distributions with extended covariance properties," *J. Math. Phys.*, vol. 33, no. 7, pp. 2515–2527, 1992.

[9] P. O. Amblard and J. L. Lacoume, "Construction of fourth order Cohen's class: A deductive approach," in *Proc. IEEE Int. Symp. Time–Freq. Time-Scale Anal.*, 1992, pp. 257–260.

[10] P. O. Amblard, "Statistiques d'ordre supérieur pour les signaux non gaussiens, non lineaires, non stationnaires," Ph.D. dissertation, Univ. Joseph Fourier, Grenoble, France, 1994.

[11] L. J. Stanković, "A multitime definition of the Wigner higher order distribution: L-Wigner distribution," *IEEE Signal Processing Lett.*, vol. 1, pp. 106–109, July 1994.

[12] ——, "A method for improved distribution concentration in the time-frequency analysis of multicomponent signals using the L-Wigner distribution," *IEEE Trans. Signal Processing*, vol. 43, pp. 1262–1268, May 1995.

[13] B. Boashash and P. O'Shea, "Polynomial Wigner–Ville distributions and their relationship to time-varying higher order spectra," *IEEE Trans. Signal Processing*, vol. 42, pp. 216–220, Jan. 1994.

[14] B. Boashash and B. Ristic, "Polynomial time-frequency distributions and time-varying higher-order spectra: Application to the analysis of multicomponent FM signals and to the treatement of multiplicative noise," *Signal Process.*, vol. 67, no. 1, 1998.

[15] F. Auger and P. Flandrin, "Improving the readability of time-frequency and time-scale representations by the reassignment method," *IEEE Trans. Signal Processing*, vol. 43, pp. 1068–1089, May 1995.

[16] R. G. Baraniuk and D. L. Jones, "Signal-dependent time-frequency analysis using a radially Gaussian kernel," *Signal Process.*, vol. 32, no. 2, pp. 263–284, June 1993.

[17] P. Loughlin, J. Pitton, and L. Atlas, "Construction of positive time-frequency distributions," *IEEE Trans. Signal Processing*, vol. 42, pp. 2697–2705, Oct. 1994.

[18] J. C. O'Neill and P. Flandrin, "Two excursions into the quartic domain," in *Proc. IEEE Int. Symp. Time–Freq. Time-Scale Anal.*, 1998, pp. 321–324.

[19] J. C. O'Neill and W. J. Williams, "A function of time, frequency, lag, and doppler," *IEEE Trans. Signal Processing*, vol. 47, pp. 789–799, Mar. 1999.

[20] ——, "Quadralinear time-frequency representations," *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, vol. 2, pp. 1005–1008, 1995.

[21] J. C. O'Neill, "A function of time, frequency, lag, and doppler with applications to kernel design," in *Proc. SPIE—Adv. Signal Process. Algorithms*, 1997.

[22] W. M. Siebert, "Studies of Woodward's uncertainty function," in *Quart. Progress Rep., Electron. Res. Lab.*. Cambridge, MA: Mass. Inst. Technol., Apr. 15, 1958, pp. 90–94.

[23] C. A. Stutt, "Some results on real-part/imaginary-part and magnitude-phase relations in ambiguity functions," *IEEE Trans. Inform. Theory*, vol. IT-10, pp. 321–327, Oct. 1965.

[24] A. J. E. M. Janssen, "On the locus and spread of psuedo-density functions in the time-frequency plane," *Philips J. Res.*, vol. 37, no. 3, pp. 79–110, 1982.

[25] H. H. Szu and H. J. Caulfield, "The mutual time-frequency content of two signals," *Proc. IEEE*, vol. 72, pp. 902–908, July 1984.

[26] B. Boashash, "Interpreting and estimating the instantaneous frequency of a signal—Part I: Fundamentals; Part II: Algorithms and applications," *Proc. IEEE*, vol. 80, pp. 519–568, Apr. 1992.

[27] P. Loughlin and B. Tracer, "Instantaneous frequency and the conditional mean frequency of a signal," *Signal Process.*, vol. 60, pp. 153–162, 1997.

[28] S. Barbarossa, "Analysis of multicomponent LFM signals by a combined Wigner–Hough transform," *IEEE Trans. Signal Processing*, vol. 43, pp. 1511–1515, June 1995.

[29] S. Kay and G. F. Boudreaux-Bartels, "On the optimality of the Wigner distribution for detection," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, 1985, pp. 1017–1020.

[30] B. Ristic and B. Boashash, "Kernel design for time-frequency signal analysis using the Radon transform," *IEEE Trans. Signal Processing*, vol. 41, pp. 1996–2008, May 1993.

[31] J. C. Wood and D. T. Barry, "Tomographic time-frequency analysis and its application toward time-varying filtering and adaptive kernel design for multicomponent linear-FM signals," *IEEE Trans. Signal Processing*, vol. 42, pp. 2094–2104, Aug. 1994.

[32] ——, "Radon transformation of time-frequency distributions for analysis of multicomponent signals," *IEEE Trans. Signal Processing*, vol. 42, pp. 3166–3177, Nov. 1994.

[33] W. Wang, A. K. Chan, and C. K. Chui, "Linear frequency-modulated signal detection using Radon-ambiguity transform," *IEEE Trans. Signal Processing*, vol. 46, pp. 571–586, Mar. 1998.

[34] J. C. O'Neill and P. Flandrin, "Chirp hunting," in *Proc. IEEE Int. Symp. Time–Freq. Time-Scale Anal.*, 1998, pp. 425–428.

[35] L. Cohen, "Generalized phase-space distribution functions," *J. Math. Phys.*, vol. 7, pp. 781–786, 1966.

[36] P. J. Loughlin, J. W. Pitton, and L. E. Atlas, "Bilinear time-frequency representaions: New insights and properties," *IEEE Trans. Signal Processing*, vol. 41, pp. 750–767, Feb. 1993.

[37] F. Auger, "Représentations temps-fréquenc des signaux non-stationnaires: Synthèse et contribution," Ph.D. dissertation, Univ. Nantes, Nantes, France, 1991.

[38] A. J. E. M. Janssen, "Positivity and spread of bilinear time-frequency distributions," in *The Wigner Distribution—Theory and Applications*, W. Mecklenbräuker and F. Hlawatsch, Eds. Amsterdam, The Netherlands: Elsevier, 1998.

[39] J. R. Fonollosa and C. L. Nikias, "Wigner higher order moment spectra: Definition, properties, computation, and application to transient signal analysis," *IEEE Trans. Signal Processing*, vol. 41, pp. 245–266, Jan. 1993.

[40] R. L. Murray, A. Papandreou-Suppappola, and G. F. Boureaux-Bartels, "New time-frequency representations: Higher order warped Wigner distributions," in *Proc. 31st Asilomar Conf. Signals, Syst., Comput.*, Pacific Grove, CA, Oct. 1997.

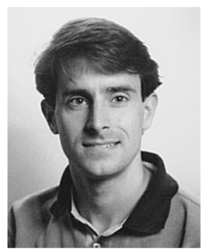
[41] L. J. Stanković, "On the realization of the polynomial Wigner–Ville distirbution for multicomponent signals," *IEEE Signal Processing Lett.*, vol. 5, pp. 157–159, July 1998.

[42] R. G. Shenoy and T. W. Parks, "The Weyl correspondence and time-frequency analysis," *IEEE Trans. Signal Processing*, vol. 42, pp. 318–331, Feb. 1994.

[43] R. G. Baraniuk, "Covariant time-frequency representations through unitary equivalence," *IEEE Signal Processing Lett.*, vol. 3, pp. 79–81, Mar. 1996.

[44] F. Hlawatsch and P. Flandrin, "The interference structure of the Wigner distribution and related time-frequency signal representations," in *The Wigner Distribution—Theory and Applications in Signal Processing*, W. Mecklenbräuker and F. Hlawatsch, Eds. Amsterdam, The Netherlands: Elsevier, 1998.

[45] P. Flandrin and P. Gonçalvès, "Geometry of affine time-frequency representations," *Appl. Comp. Harmon. Anal.*, vol. 3, no. 1, pp. 10–39, 1996.



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