Shift Covariant Time–Frequency Distributions of Discrete Signals

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***Abstract*—Many commonly used time–frequency distributions are members of the Cohen class. This class is defined for continuous signals, and since time–frequency distributions in the Cohen class are quadratic, the formulation for discrete signals is not straightforward. The Cohen class can be derived as the class of all quadratic time–frequency distributions that are covariant to time shifts and frequency shifts. In this paper, we extend this method to three types of discrete signals to derive what we will call the discrete Cohen classes. The properties of the discrete Cohen classes differ from those of the original Cohen class. To illustrate these properties, we also provide explicit relationships between the classical Wigner distribution and the discrete Cohen classes.**

I. Introduction

In SIGNAL analysis, there are four types of signals commonly used. These four types are based on whether the signal is continuous or discrete and whether the signal is aperiodic or periodic. The four signal types are listed in Table I along with their properties in the time domain. For each of the four types of signals, there is an appropriate Fourier transform pair, so it seems plausible that there should exist four types of time-frequency distributions (TFD's). The Cohen class [1], [2] (with the restriction that the kernel is not a function of time and frequency and is also not a function of the signal) can derived axiomatically as the class of all quadratic TFD's for type I signals that are covariant to time shifts and frequency shifts [3]–[5]. In this paper, we will investigate the quadratic, time and frequency shift covariant classes of TFD's for the other three types of signals. The original class will be renamed the type I Cohen class, and the other three classes will be denoted the type II, III, and IV Cohen classes.

There are three common methods for deriving TFD's for type I signals. The first uses operator theory [1], [2], the second uses group theory [6], and the third uses covariance properties [3]–[5]. In this paper, we choose to use the covariance-based approach to investigate TFD's for signals of types II, III, and IV because of the simplicity and directness of the mathematics. Narayanan *et al.* [7], [8] have investigated the formulation of a type IV TFD's using operator theory. Richman *et al.* have investigated type IV Wigner distributions using group theory [9]. There has also been much other work investigating methods for computing TFD's from sampled signals [10]–[30]. The results presented here are more complete than the above results in that we give a closed form for the complete class of shift-covariant TFD's for signals of types II, III, and IV. Since the class of AF-GDTFD's introduced by Jeong and Williams is quadratic and shift-covariant, it is clearly a subset of the type II Cohen class; however, nothing more can be said at this point. The type IV Wigner distribution produced by Richman *et al.* [9] and the distributions produced by Narayanan *et al.* [7] are members of the type IV Cohen class, but they have not generated a class of type IV distributions.

This paper is organized as follows. Section II presents some basic characteristics of TFD's for each of the four signal types. Section III repeats a derivation of the type I Cohen class as the class of time and frequency shift covariant, quadratic TFD's. This derivation will be extended to derive the other three Cohen classes. Sections IV–VI will present results concerning the three discrete Cohen classes, and Section VII will present some practical issues regarding the computation of TFD's.

TABLE I
Definitions of the Four Types of Signals

Type	Time Domain Properties	Spectrum
type I	continuous and aperiodic	Fourier transform
type II	discrete and aperiodic	discrete-time Fourier transform
type III	continuous and periodic	Fourier series
type IV	discrete and periodic	discrete Fourier transform

II. Four Signal Types

The characteristics of the four types of signals in the time and frequency domains will determine the corresponding characteristics of the TFD's. Here, we discuss the characteristics of the four types of TFD's that lead to the corresponding time–frequency surfaces in Fig. 1.

A type I signal $x(t)$ will be continuous and aperiodic. The Fourier transform of this signal $X(\omega)$ will also be continuous and aperiodic. We assume that both $x(t)$ and $X(\omega)$ are square integrable and, thus, will be elements of $L_2(\mathsf{IR})$. TFD's for this type of signal will have time and frequency variables that are continuous and aperiodic, so a type I TFD $C_x^{\mathrm{I}}(t,\omega)$ will be an element of $L_2(\mathsf{IR}^2)$. The time–frequency surface for a type I TFD is a plane. The class of shift covariant TFD's for type I signals are covariant to shifts of the form

$$\left(\mathbf{T}_{t_0}^{\mathrm{I}}x\right)(t) = x(t-t_0)$$
$$\left(\mathbf{F}_{\omega_0}^{\mathrm{I}}x\right)(t) = x(t)e^{j\omega_0 t}.$$

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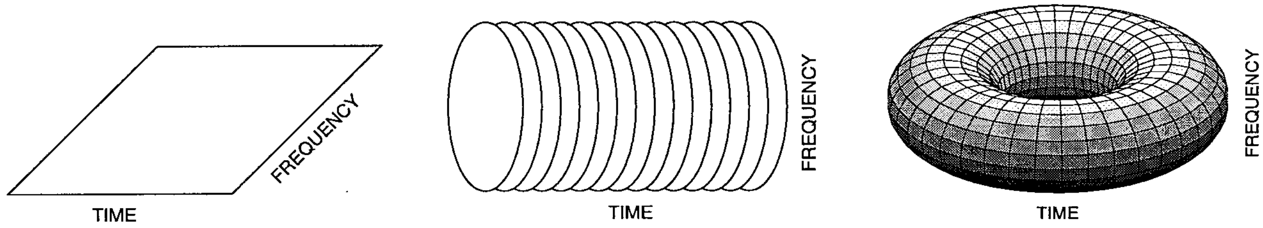


(a) (b) (c)

Fig. 1. Time–frequency surfaces for type I, II, and IV TFD's.

A type II signal $x(n)$ will be discrete and aperiodic. The discrete-time Fourier transform of this signal $X(\omega)$ will be continuous and periodic. We assume that $x(n)$ is an element of $\ell_2(\mathbb{Z})$ and that $X(\omega)$ is an element of $L_2([0, 2\pi))$. TFD's for this type of signal will have a discrete, aperiodic time variable and a continuous, periodic frequency variable; therefore, a type II TFD $C_x^{\mathrm{II}}(n, \omega)$ will be a countably infinite collection of elements of $L_2([0, 2\pi))$. Since the frequency variable of a type II TFD is periodic, the time–frequency surface will be slices of a cylinder. The class of shift covariant TFD's for type II signals will be covariant to shifts of the form

$$\left(\mathbf{T}_{n_0}^{\mathrm{II}} x\right)(n) = x(n - n_0)$$
$$\left(\mathbf{F}_{\omega_0}^{\mathrm{II}} x\right)(n) = x(n) e^{j\omega_0 n}.$$

A type III signal is the dual of a type II signal, so a type III TFD will be the dual of a type II TFD.

A type IV signal $x(n)$ will be discrete and periodic with period N. The discrete Fourier transform $X(k)$ will also be discrete and periodic with period N. We assume that both $x(n)$ and $X(k)$ are elements of $\ell_2([1, N])$. TFD's for this type of signal will have time and frequency variables that are discrete and periodic, so a type IV TFD $C_x^{\mathrm{IV}}(n, k)$ will be a member of $\ell_2([1, N]^2)$. Since the time and frequency variables of a type IV TFD are periodic, the time–frequency surface will be points on a torus. The class of shift covariant TFD's for type IV signals will be covariant to shifts of the form

$$\left(\mathbf{T}_{n_0}^{\mathrm{IV}} x\right)(n) = x(n - n_0)$$
$$\left(\mathbf{F}_{k_0}^{\mathrm{IV}} x\right)(n) = x(n) e^{j2\pi n k_0/N}.$$

III. Shift-Covariant TFD's

Here, we will repeat a derivation that shows that the Cohen class (with kernels that are independent of the signal and also independent of time and frequency) is the class of quadratic, time and frequency shift-covariant TFD's [3]–[5]. This concept will be adapted to derive the discrete Cohen classes. The most general form for a bilinear function of two type I signals $x(t)$ and $y(t)$ is[1]

$$T_{x,y}(\underline{\Lambda}) = \iint k(\underline{\Lambda}; t_1, t_2) x(t_1) y^*(t_2)\, dt_1\, dt_2$$

where $\underline{\Lambda}$ is some quantity of interest. If we let $x(t) = y(t)$, and let $\underline{\Lambda} = [t, \omega]$, then we have the most general quadratic TFD of a type I signal $x(t)$.

$$T_x(t, \omega) = \iint k(t, \omega; t_1, t_2) x(t_1) x^*(t_2)\, dt_1\, dt_2. \tag{1}$$

For the signal $x(t)$, define a shifted version in time and frequency as

$$\tilde{x}(t) = \left(\mathbf{T}_{t_0}^{\mathrm{I}} \mathbf{F}_{\omega_0}^{\mathrm{I}} x\right)(t).$$

If it is desired that the TFD be covariant to time and frequency shifts, then it must be true that

$$T_{\tilde{x}}(t, \omega) = T_x(t - t_0, \omega - \omega_0).$$

Under the above constraint, (1) simplifies to the well-known Cohen class of TFD's. We present the Cohen class in four different forms:

$$C_x^{\mathrm{I}}(t, \omega) = \iint x(t_1) x^*(t_2) \psi(t_1 - t, t_2 - t) e^{-j\omega(t_1 - t_2)}\, dt_1\, dt_2 \tag{2a}$$

$$= \iint R_x^{\mathrm{I}}(t', \tau) \phi(t - t', \tau) e^{-j\omega\tau}\, dt'\, d\tau \tag{2b}$$

$$= \frac{1}{2\pi} \iint X(\omega_1) X^*(\omega_2) \Psi(\omega_1 - \omega, \omega_2 - \omega) \times e^{jt(\omega_1 - \omega_2)}\, d\omega_1\, d\omega_2 \tag{2c}$$

$$= \frac{1}{2\pi} \iint R_X^{\mathrm{I}}(\theta, \omega') \Phi(\theta, \omega - \omega') e^{jt\theta}\, d\omega'\, d\theta \tag{2d}$$

where the type I temporal local auto correlation function (LACF) R_x^{I} and the type I spectral LACF R_X^{I} are defined as

$$R_x^{\mathrm{I}}(t, \tau) = x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right)$$
$$R_X^{\mathrm{I}}(\theta, \omega) = X\left(\omega + \frac{\theta}{2}\right) X^*\left(\omega - \frac{\theta}{2}\right)$$

and the kernel functions are related by

$$\psi(t_1, t_2) = \phi\left(\frac{-t_1 - t_2}{2}, t_1 - t_2\right) \tag{3a}$$

$$\Psi(\omega_1, \omega_2) = \frac{1}{2\pi} \iint \psi(t_1, t_2) e^{j\omega_1 t_1} e^{-j\omega_2 t_2}\, dt_1 dt_2 \tag{3b}$$

$$\Phi(\theta, \omega) = \iint \phi(t, \tau) e^{-j\theta t} e^{-j\omega\tau}\, dt\, d\tau. \tag{3c}$$

The two forms in (2a) and (2c) arrive naturally from the derivation and will allow simpler notation for the discrete Cohen classes, as will be seen below. The two forms in (2b) and (2d) are more commonly used and allow simpler notation for several kernel constraints.

[1] Unless otherwise indicated, the range of sums and integrals will be assumed to be $-\infty$ to ∞.

It is interesting to compare (2a) with the equation for the type I spectrogram

$$S_x^{\mathrm{I}}(t,\omega) = \iint x(t_1)x^*(t_2)h(t_1-t)h^*(t_2-t)e^{-j\omega(t_1-t_2)}\,dt_1\,dt_2.$$

The only difference between the two equations is that the outer product of the spectrogram window is replaced by the kernel in the Cohen class. As a result, the Cohen class can be considered to be a generalization of the spectrogram where a two-dimensional (2-D) function of rank one (the outer product of the spectrogram window) is replaced by a 2-D function of arbitrary rank (the kernel). This form makes it clear that the spectrogram is a member of the Cohen class and that, under certain constraints, elements of the Cohen class can be decomposed into weighted sums of spectrograms [31].

The cross terms in the Wigner distribution [32] satisfy the following properties.

- Cross terms are centered exactly between two auto terms.
- If two auto terms are separated in frequency by Δ_ω, then the rate of oscillation of the cross term in the time direction is Δ_ω.
- If two auto terms are separated in time by Δ_t, then the rate of oscillation of the cross term in the frequency direction is Δ_t.

If we constrain the kernel such that $\phi(t,\tau)=\phi^*(-t,\tau)$, then the representation of the kernel in the ambiguity plane will be real,[2] and the cross terms of the corresponding TFD will also have the properties indicated above. Other distributions in the Cohen class, such as the Rihaczek distribution, whose kernels do not satisfy the above constraint will not have the above cross term properties.

TFD's in the Cohen class are 2-D, continuous functions. As a means for representing these functions, we can compute samples of these 2-D functions such that the continuous function could be recovered through sinc interpolation [23], [24]. The method presented in [23] and [24] is unnecessarily complicated, and a simpler method that uses an oversampled signal is presented in [33]. Note that these methods only provide accurate results when the *kernel* is bandlimited (and thus can be sampled without aliasing) and has a closed form in the time–frequency domain.

IV. The Type II Cohen Class

The above proof for the type I Cohen class extends directly to form the type II Cohen class, which is identical to the class of AF-GDTFD's [21]. The AF-GDTFD's were known to be covariant to time and frequency shifts, but it was not known until this point that the AF-GDTFD's include all type II TFD's that are covariant to time and frequency shifts. To eliminate the clumsy acronym and to emphasize the analogy with the original Cohen class, we will rename the class of AF-GDTFD's as the type II Cohen class. We will present the type II Cohen class in four different forms:

$$C_x^{\mathrm{II}}(n,\omega) = \sum_{n_1}\sum_{n_2} x(n_1)x^*(n_2)\psi(n_1-n,n_2-n)e^{-j\omega(n_1-n_2)} \tag{4a}$$

$$= \sum\sum_{\{n'\pm\frac{m}{2},m\}\in\mathbb{Z}} R_x^{\mathrm{II}}(n',m)\,\phi(n-n',m)e^{-j\omega m} \tag{4b}$$

$$= \frac{1}{2\pi}\iint_0^{2\pi} X(\omega_1)X^*(\omega_2)\Psi(\omega_1-\omega,\omega_2-\omega) \times e^{jn(\omega_1-\omega_2)}\,d\omega_1\,d\omega_2 \tag{4c}$$

$$= \frac{1}{2\pi}\iint_{0<\omega'\pm\frac{\theta}{2}<2\pi} R_X^{\mathrm{II}}(\theta,\omega')\Phi(\theta,\omega-\omega')e^{jn\theta}\,d\omega'\,d\theta \tag{4d}$$

where the type II temporal LACF and spectral LACF are defined as

$$R_x^{\mathrm{II}}(n,m) = x\Big(n+\frac{m}{2}\Big)x^*\Big(n-\frac{m}{2}\Big)$$
$$R_X^{\mathrm{II}}(\theta,\omega) = X\Big(\omega+\frac{\theta}{2}\Big)X^*\Big(\omega-\frac{\theta}{2}\Big)$$

and the kernels are related to each other analogous to (3). An unusual feature of $R_x^{\mathrm{II}}(n,m)$ and $\phi(n,m)$ is that they are only defined when $\{n\pm\frac{m}{2},m\}\in\mathbb{Z}$, resulting in a hexagonal sampling grid. The class of AF-GDTFD's were originally presented [21] in the form of (4b); however, this notation is somewhat cumbersome due to the hexagonal sampling. The forms in (4a) and (4c) arrive naturally from the derivation and provide a more elegant notation.

The type II Cohen class can also be considered to be a generalization of the type II spectrogram

$$S_x^{\mathrm{II}}(n,\omega) = \sum_{n_1}\sum_{n_2} x(n_1)x^*(n_2)h(n_1-n)h^*(n_2-n)e^{-j\omega(n_1-n_2)}.$$

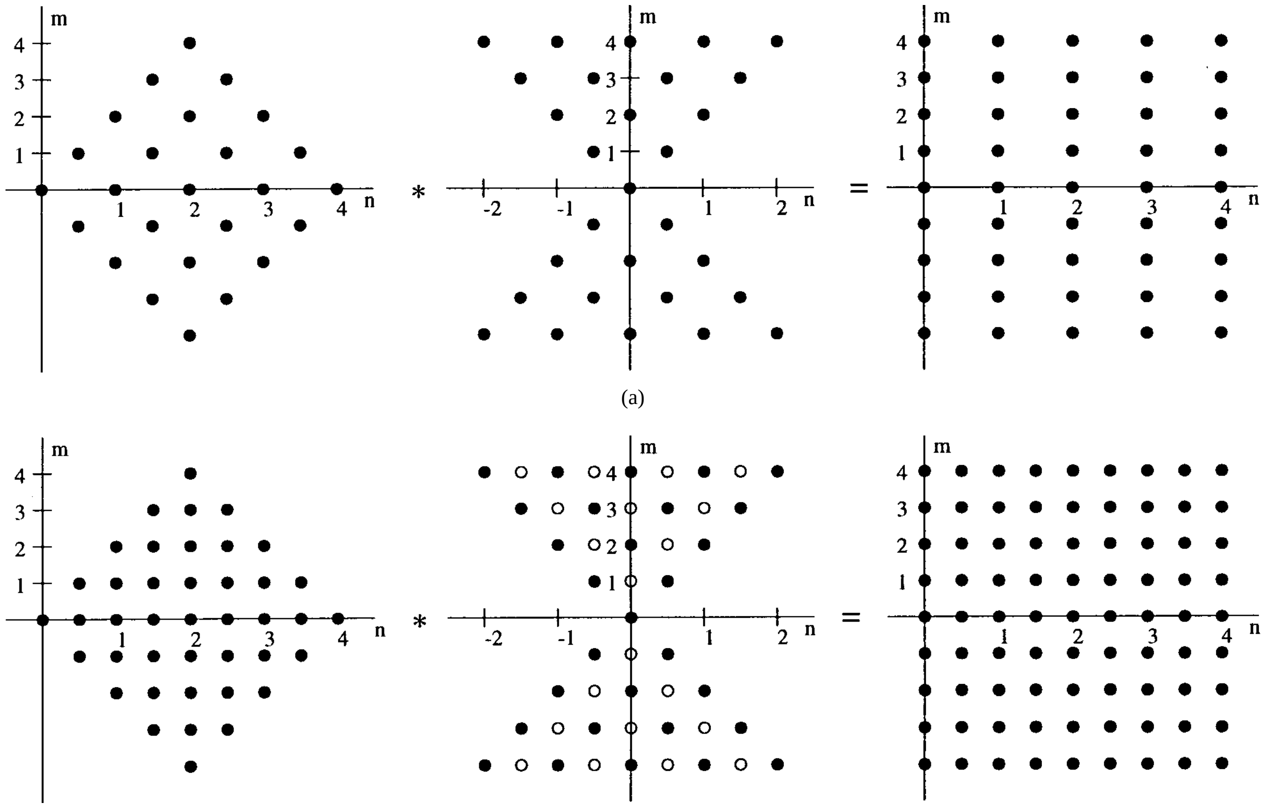
This form makes it clear that the type II spectrogram is a member of the type II Cohen class and that elements in the type II Cohen class can be decomposed into a weighted sum of type II spectrograms [34].

A. Distributions in the Type II Cohen Class

In Table II, we present the kernels of several time–frequency distributions in the Cohen class. The kernels are formulated in the time-lag plane since this form of the kernel is discrete in both variables. Note that the kernels are defined on a hexagonal sampling grid as indicated above.

The kernels corresponding to the spectrogram, Born-Jordan, Rihaczek, Page, and Levin distributions are all direct discretizations of the corresponding kernels for the type I Cohen class TFD's [2], [3], [32]. The binomial kernel [35] satisfies many desirable properties of TFD's, and the recursive structure also allows the implementation of fast algorithms. The type II Cohen class also provide the framework for the discrete formulation of the cone kernel [29]. These discrete TFD's are all computed from the Nyquist sampled signal, are covariant to time shifts and circular frequency shifts, and satisfy many of the properties of the corresponding type I Cohen class TFD's, e.g., marginals. For example, the type I and type II Rihaczek

[2] The kernel operates on the Wigner distribution as a linear, time-invariant filter. The frequency response of this filter is the ambiguity plane representation of the kernel; therefore, if this representation is real, then the phase of the filter is either 0 or π.



(a) (b)

Fig. 2. Equivalent methods for computing TFD's in the type II Cohen class. On the left is the LACF, in the middle is the kernel, and on the right is the generalized LACF. Open circles represent zero values, and filled circles represent actual or interpolated values.

TABLE II
SOME KERNELS FOR THE TYPE II COHEN CLASS

Spectrogram:	$\phi(n,m) = h(n+\frac{m}{2})\, h^*(n-\frac{m}{2}).$
Binomial:	$\phi(n,m) = \begin{cases} \delta(n) & \text{for } m=0 \\ \frac{1}{2}[\phi(n+\frac{1}{2},m-1)+\phi(n-\frac{1}{2},m-1)] & \text{for } m>0 \\ \phi(n,-m) & \text{for } m<0 \end{cases}$
Quasi-Wigner:	$\phi(n,m) = \begin{cases} \delta(n) & \text{for even } m \\ \frac{1}{2}\delta(n+\frac{1}{2})+\frac{1}{2}\delta(n-\frac{1}{2}) & \text{for odd } m \end{cases}$
Born-Jordan:	$\phi(n,m) = \begin{cases} \frac{1}{\lvert m+1\rvert} & \text{for } \lvert n\rvert \le \frac{\lvert m\rvert}{2} \\ 0 & \text{otherwise} \end{cases}$
Rihaczek:	$\phi(n,m) = \delta(n-\frac{m}{2})$
Page:	$\phi(n,m) = \delta(n-\frac{\lvert m\rvert}{2})$
Levin:	$\phi(n,m) = \delta(n+\frac{\lvert m\rvert}{2})$
Claasen-Mecklenbräuker:	$\phi(n,m) = \begin{cases} \delta(n) & \text{for even } m \\ 0 & \text{for odd } m \end{cases}$

distributions can be expressed as

$$\mathrm{RD}_x^{\mathrm{I}}(t,\omega) = x^*(t)X(\omega)e^{-j\omega t}$$
$$\mathrm{RD}_x^{\mathrm{II}}(n,\omega) = x^*(n)X(\omega)e^{-j\omega n}.$$

A prominent distribution that is missing from the list in Table II is a type II Wigner distribution. Discretization methods [15]–[17], [28] have failed to produce a satisfactory type II Wigner distribution since they require the signal to be oversampled by a factor of two. In [36] and [37], we present an alternative definition of the type I (classical) Wigner distribution that generalizes straightforwardly to all four signal types. Under this definition, we have shown that the type II Wigner distribution does not exist. For this reason, the type II quasi-Wigner distribution was created [21]. The type II quasi-Wigner distribution provides very little smoothing, so it is, in a sense, "close" to a Wigner distribution.[3] The type II quasi-Wigner distribution will be useful for illustrating the properties of the type II Cohen class.

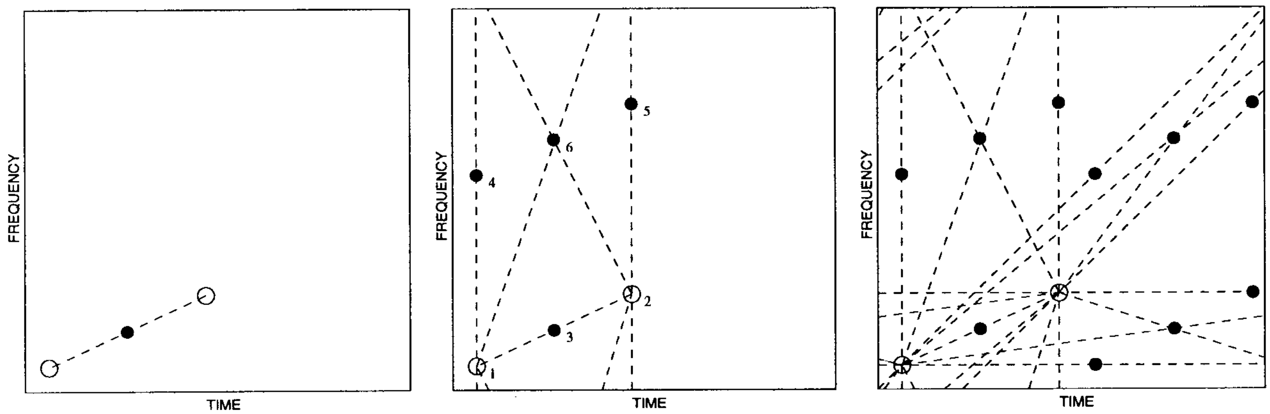
The Claasen–Mecklenbräuker (CM) distribution is equivalent to their discrete implementation of the type I Wigner distribution [17]. The CM distribution is related to a scaled and sampled version of the type I Wigner distribution when the signal is oversampled by a factor of two. However, since the CM distribution requires the signal to be oversampled by two, it should not be considered a type II Wigner distribution.

B. Relationship with the Classical Wigner Distribution

Although the type II Cohen class is equivalent to the class of AF-GDTFD's, the properties of the type II Cohen class are not well understood. In particular, TFD's in the type II Cohen class appear to have more terms in the time–frequency plane than TFD's in the type I Cohen class. It is unclear what these components represent, and authors have attributed them as due to aliasing [18], [22]. In this section, we will present a means for computing type II TFD's from the type I (classical) Wigner distribution as a means for explaining the properties of the type II Cohen class.

The procedure for computing a TFD in the type II Cohen class is represented pictorially in Fig. 2(a). On the left is the

[3] In [21], this distribution was called, without justification, a discrete Wigner distribution. We have chosen to call it a quasi-Wigner distribution because it is "close", but it is not the real thing.



(a) (b) (c)

Fig. 3. Pictorial representation of the cross terms in the type I, II, and IV Cohen classes. Open circles represent auto terms, and filled circles represent cross terms. The dashed lines show how the cross terms are between the auto terms.

hexagonally sampled LACF, in the middle is the hexagonally sampled kernel, and on the right is the generalized LACF. To compute the TFD, we would perform discrete-time Fourier transforms on the lag variable of the generalized LACF. In Fig. 2(b), we present an alternative method for computing TFD's in the type II Cohen class by means of the classical Wigner distribution. To do this, we double the number of points in the LACF with sinc interpolation and double the number points in the kernel by inserting zeros. If we denote the modified LACF as $\tilde{R}_x^{\mathrm{II}}(n,m)$ and the modified kernel by $\tilde{\phi}(n,m)$, then it is straightforward to see that we are computing the exact same distribution

$$C_x^{\mathrm{II}}(n,\omega) = \sum_{\{2n',m\}\in\mathbb{Z}}\sum \tilde{R}_x^{\mathrm{II}}(n',m)\,\tilde{\phi}(n-n',m)e^{-j\omega m}\bigg|_{n\in\mathbb{Z}}.$$

However, if we reverse the order of the summations, then we can express TFD's in the type II Cohen class in terms of a time-sampled, type I Wigner distribution[4]

$$C_x^{\mathrm{II}}(n,\omega) = W_x^{\mathrm{I}}(n,\omega) *_n \circledast_\omega \mathcal{P}(n,\omega)\big|_{n\in\mathbb{Z}}$$

where

$$\mathcal{P}(n,\omega) = \sum_m \tilde{\phi}(n,m)e^{-j\omega m}$$

and $W_x^{\mathrm{I}}(n,\omega)$ represents a time-sampled, type I Wigner distribution. For more details, see [33].

Three examples of modified kernels in the time–frequency plane $\mathcal{P}(n,\omega)$ are shown in Fig. 4. The first corresponds to the type II binomial distribution, the second corresponds to the type II quasi-Wigner distribution, and the third corresponds to a type II spectrogram with a Hanning window. Note that since all three kernels satisfy $\phi(n,m) = \phi^*(-n,m)$ analogous to the restriction above for the type I Cohen class, the phase of these filters is 0 or π.

Each kernel contains two distinct parts. The smooth part of each kernel is what we would expect and will be called the lowpass part of the kernel. The oscillating part is centered at a frequency of π rad and will be called the highpass part of the kernel. TFD's in the type II Cohen class are computed by performing a 2-D convolution of $W_x^{\mathrm{I}}(n,\omega)$ with $\mathcal{P}(n,\omega)$. The lowpass part of each kernel will perform as expected by capturing elements of the Wigner distribution that are slowly varying. The highpass part of the kernel will capture elements of the Wigner distribution that are quickly varying and displace them in frequency by π rad. The highpass part of the kernel is an unexpected, but integral, part of the type II Cohen class. The highpass part of the kernel cannot be eliminated and is what makes the properties of the type II Cohen class different from the properties of type I Cohen class. In fact, the highpass part of the kernel is necessary if the TFD is to satisfy the frequency shift covariance property [33].

TABLE III
AUTO TERMS AND CROSS TERMS IN A TYPE II TFD OF A TWO COMPONENT SIGNAL

	component	location	type
1	$W_x^I(n,\omega) *_t \circledast_\omega \mathcal{P}_L(n,\omega)$	(n_1,ω_1)	auto term
2	$W_y^I(n,\omega) *_t \circledast_\omega \mathcal{P}_L(n,\omega)$	(n_2,ω_2)	auto term
3	$2\Re\{W_{x,y}^I(n,\omega) *_t \circledast_\omega \mathcal{P}_L(n,\omega)\}$	$(\frac{n_1+n_2}{2}, \frac{\omega_1+\omega_2}{2})$	cross term
4	$W_x^I(n,\omega) *_t \circledast_\omega \mathcal{P}_H(n,\omega)$	$(n_1,\omega_1+\pi)$	cross term
5	$W_y^I(n,\omega) *_t \circledast_\omega \mathcal{P}_H(n,\omega)$	$(n_2,\omega_2+\pi)$	cross term
6	$2\Re\{W_{x,y}^I(n,\omega) *_t \circledast_\omega \mathcal{P}_H(n,\omega)\}$	$(\frac{n_1+n_2}{2}, \frac{\omega_1+\omega_2}{2}+\pi)$	cross term

As a means for illustrating the properties of the type II Cohen class, we will now apply the above to a fictional, two-component signal with the first component centered at (n_1,ω_1) and the second component centered at (n_2,ω_2). The Wigner distribution of this signal is represented pictorially in Fig. 3(a), where the open circles represent the two auto terms, and the filled circle represents the cross term. Since there are two parts to the modified kernel, the type II TFD will have six terms rather than three. These six terms are are represented pictorially in Fig. 3(b) and are listed in Table III, where $\mathcal{P}_L(n,\omega)$ and $\mathcal{P}_H(n,\omega)$ represent, respectively, the lowpass and highpass parts of the modified kernel. The first two terms represent the signal components and will be called auto terms. The last four terms are centered between the two auto terms on the cylinder and will be called cross terms.

[4] $*_n$ denotes a convolution in the variable n, and $\circledast_\omega$ denotes a circular convolution in the variable ω.

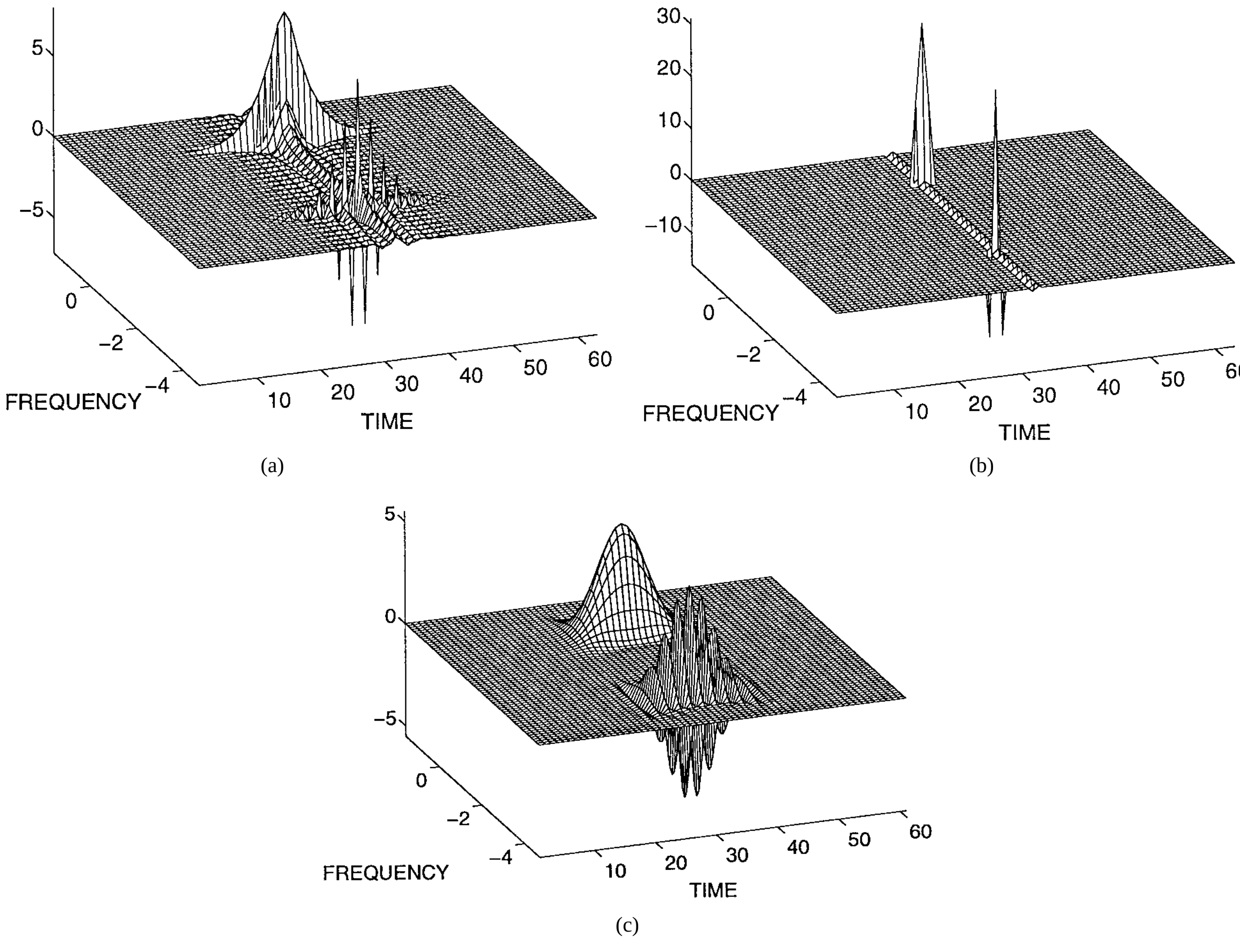


Fig. 4. Three modified kernels for the type II Cohen class in the time–frequency plane.

Cross term four [cf., Fig. 3(b)] lies between auto term one and the periodic repetition of itself. This cross term arrives from filtering autoterm one with the highpass part of the kernel. Thus, like the auto term one, cross term four does not oscillate. However, since cross term four is being filtered by the highpass part of the kernel, it is attenuated *as if it were oscillating in time at a rate of* 2π *rad and being attenuated by the lowpass part of the kernel*. This equivalence is a result of the fact that the lowpass and highpass parts of the kernel are related by a simple modulation. By removing the modulation from the highpass part of the kernel and transferring it to the nonoscillating auto term, we can see that this is equivalent to lowpass filtering a highly oscillatory term. Cross term five is analogous to cross term four.

Cross terms three and six arise from filtering the cross term in the Wigner distribution, are centered on the cylinder between the two auto terms, and will have the same frequency of oscillation. However, since cross term three is being filtered by the lowpass part of the kernel and cross term six is being filtered by the highpass part of the kernel, the cross terms will *not be attenuated by the same amount*.[5] Even though the two terms have the same rate of oscillation in the time direction, the term that is closest to the auto terms will be attenuated the most. This is discussed in much greater detail in [33].

In general, the cross terms in the type II Cohen class satisfy the following properties, which are analogous to the cross term properties of the type I Cohen class.[6]

- Cross terms are centered exactly between two auto terms, where “between” is applied on the surface of the cylinder.
- If two auto terms are separated in time by n_0, then the rate of oscillation of the cross term in the frequency direction will be n_0.
- If two auto terms are separated in frequency by ω_0, then the rate of oscillation of the cross term in the time direction will be $\min\{\omega_0, 2\pi - \omega_0\}$.
- The ability of the kernel to attenuate the cross term is directly related to the distance between the cross term and the corresponding auto terms (regardless of the rate of oscillation of the cross term).

C. Examples

For the first example, we use sinusoids of three different frequencies. The binomial distributions of these sinusoids are shown in Fig. 5(a)–(c). For the low-frequency sinusoid, the Wigner distribution will have two auto terms and a slowly varying cross term between them. Since the three components in the Wigner distribution are all slowly varying, the three

[5] Unless the two auto terms are separated by π rad in frequency. In this case, the two auto terms will be attenuated by exactly the same amount.

[6] Subject to the kernel constraint mentioned above.

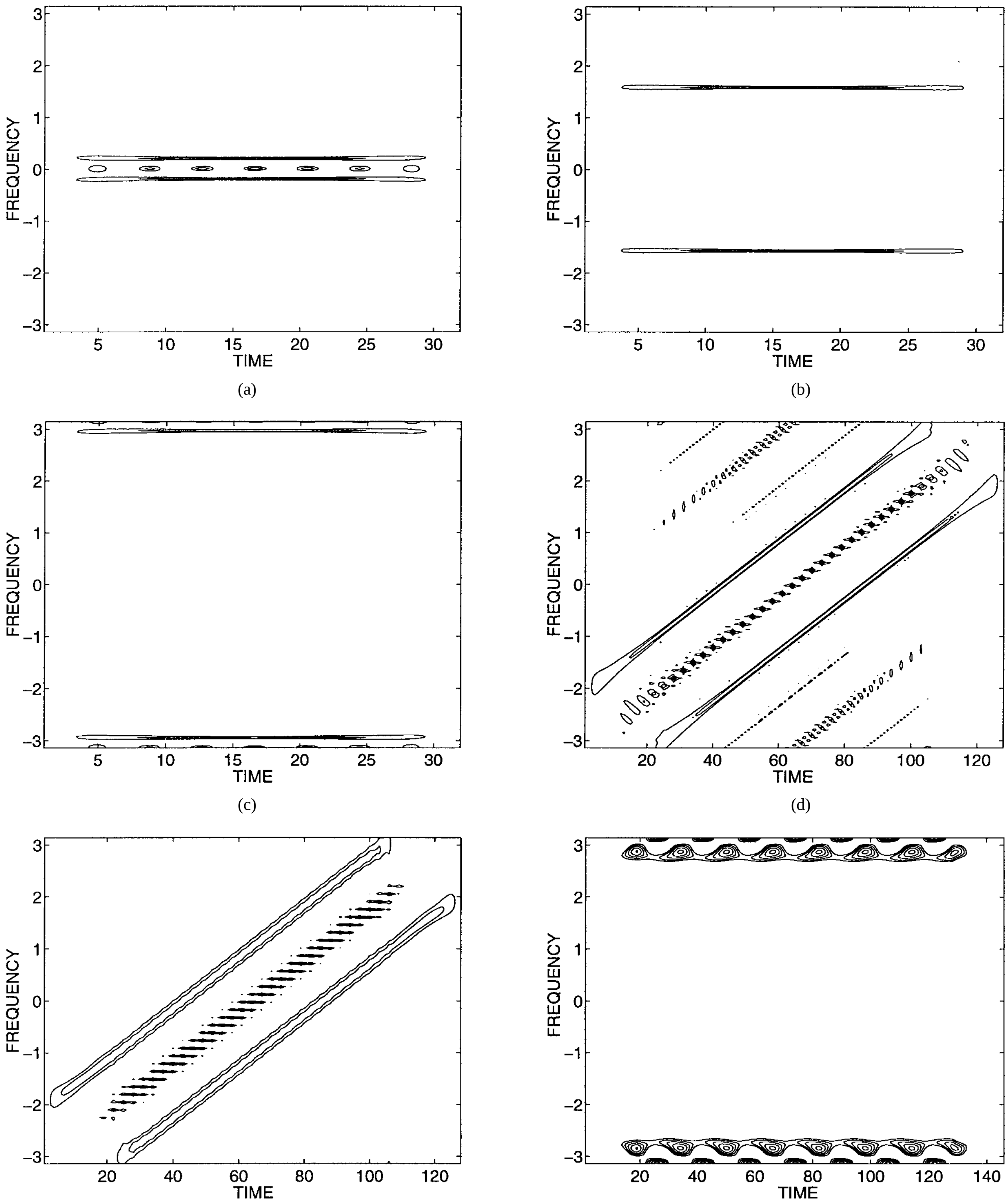


Fig. 5. Examples illustrating the properties of the type II Cohen class.

components are all captured by the lowpass part of the kernel and ignored by the highpass part of the kernel.

For the middle frequency sinusoid, the Wigner distribution will have two auto terms and a medium varying cross term. The auto terms are slowly varying and again captured by the lowpass part of the kernel. The cross term is varying too quickly for the lowpass part of the kernel and too slowly for the highpass part of the kernel and is thus captured by neither. Again, the highpass part of the kernel has no effect.

For the high-frequency sinusoid, the Wigner distribution will have two auto terms and a quickly varying cross term. As before, the auto terms are captured by the lowpass part of the kernel. However, now, the cross term is varying quickly enough to be picked up by the highpass part of the kernel and, thus, appears at a frequency of π rad.

For the second example, we will compute the quasi-Wigner distribution and binomial distribution of two parallel chirps. These TFD's are shown in Fig. 5(d) and (e). The quasi-Wigner kernel performs very little smoothing on the Wigner distribution. As a result, all the components of the Wigner distribution are captured by both parts of the quasi-Wigner kernel, and the resulting distribution has six components. Since the binomial kernel provides more smoothing than the quasi-Wigner kernel, the highpass part of the binomial kernel will have very little effect, and the binomial distribution appears to be very similar to TFD's in the type I Cohen class.

Since the type II spectrogram is a member of the type II Cohen class, the spectrogram kernel will also contain lowpass and a highpass parts. The type II spectrogram of a high-frequency sinusoid is computed using a rectangular window of length 19 and shown in Fig. 5(f). The type I spectrogram of this signal would contain two auto terms and a greatly attenuated, quickly varying cross term at a frequency of 0 rad. The highpass part of the kernel captures this cross term and creates the component at a frequency of π rad in the type II spectrogram. Due to the extreme smoothing nature of the type II spectrogram, it is difficult to find an example where all six components would be visible.

D. Additional Properties

When the class of AF-GDTFD's was originally presented by Jeong and Williams [21], they also derived kernel constraints for the distributions to satisfy many desirable properties: realness, positivity, time shift covariance, frequency shift covariance, time marginals, frequency marginals, finite time support, finite frequency support, instantaneous frequency, and group delay. However, in [21], the kernel constraints for the group delay, instantaneous frequency, and finite frequency support properties are incorrect. We have investigated this in greater detail and provided corrections in [33], but this must be omitted here due to space limitations. Here, we will briefly investigate the kernel constraints for the finite frequency support property and the Moyal formula.

For the type I Cohen class, a TFD is said to satisfy the finite frequency support property if $X(\omega)=0$ for $\omega \notin [\omega_a, \omega_b]$ implies that $C_x^{\mathrm{I}}(t,\omega)=0$ for $\omega \notin [\omega_a, \omega_b]$. However, since the frequency variable of a type II Cohen class TFD is periodic, the finite frequency support property is not well defined. A type I TFD is said to satisfy the strong frequency support property [38] if for any ω_0, $X(\omega_0)=0$ implies that $C_x^{\mathrm{I}}(t,\omega_0)=0$. This property can be satisfied, and an example of a type II TFD that satisfies this is the type II Rihaczek distribution that has been defined above.

The validity of the Moyal formula [3], [39] is useful in several applications, including signal synthesis [40] and detection/estimation problems [41]. Given two type II signals $x(n)$ and $y(n)$, the Moyal formula can be formulated for type II signals as[7]

$$\sum_n \int C_x^{\mathrm{II}}(n,\omega)\left[C_y^{\mathrm{II}}(n,\omega)\right]^* d\omega = \left|\sum_n x(n)y^*(n)\right|^2.$$

TFD's in the type II Cohen class will satisfy the Moyal formula under

$$\sum_n \phi(n,m)\phi^*(n+a,m)=\delta(a) \quad \forall m. \tag{5}$$

The proof is straightforward and is presented in [33]. Examples of TFD's that satisfy this constraint are the type II Rihaczek, Page, and Levin distributions.

E. Aliasing

The above analysis provides an explicit mechanism for understanding the properties of distributions in the type II Cohen class. The analysis clearly shows that distributions in the type II Cohen class will have more terms than distributions in the type I Cohen class. Aliasing occurs when a continuous function is sampled at a rate that is lower than the Nyquist rate. Since the operation of computing a type II Cohen class TFD is defined explicitly for type II (discrete time) signals, it is not clear how to define "aliasing" in this context. However, the following properties of TFD's in the type II Cohen class are contrary to the notion of "aliasing," and thus, we have chosen to designate the extra terms as cross terms.

- There exist distributions that satisfy the Moyal formula (and thus can reconstruct the signal to within a constant phase [33], [42]).
- The extra terms do not prevent the time and frequency marginals from being satisfied.
- The extra terms always exist, even when the signal is severely oversampled.
- The extra terms behave like cross terms with regard to their location and attenuation.
- The extra terms are necessary for the distribution to be covariant to circular frequency shifts.
- The type II spectrogram also contains these extra terms, although they are usually not apparent.

V. The Type III Cohen Class

The type III Cohen class is useful in applications such as in the analysis of scattering [43], [44], where complex frequency data is being collected and it is desired to compute TFD's of this data. However, since the type III Cohen class is the exact dual of the type II Cohen class, there is no need to investigate this class any further.

VI. The Type IV Cohen Class

There has been relatively little work investigating TFD's for type IV signals. Richman *et al.* [9] have investigated a type IV Wigner distribution using group theory. In addition, Narayanan *et al.* [7], [8] have investigated TFD's for type IV

[7]The Moyal formula can be written in a more general form that involves cross TFD's of two signals. The result given below also holds for the more general case; however, we do not wish to introduce type II cross TFD's at this point.

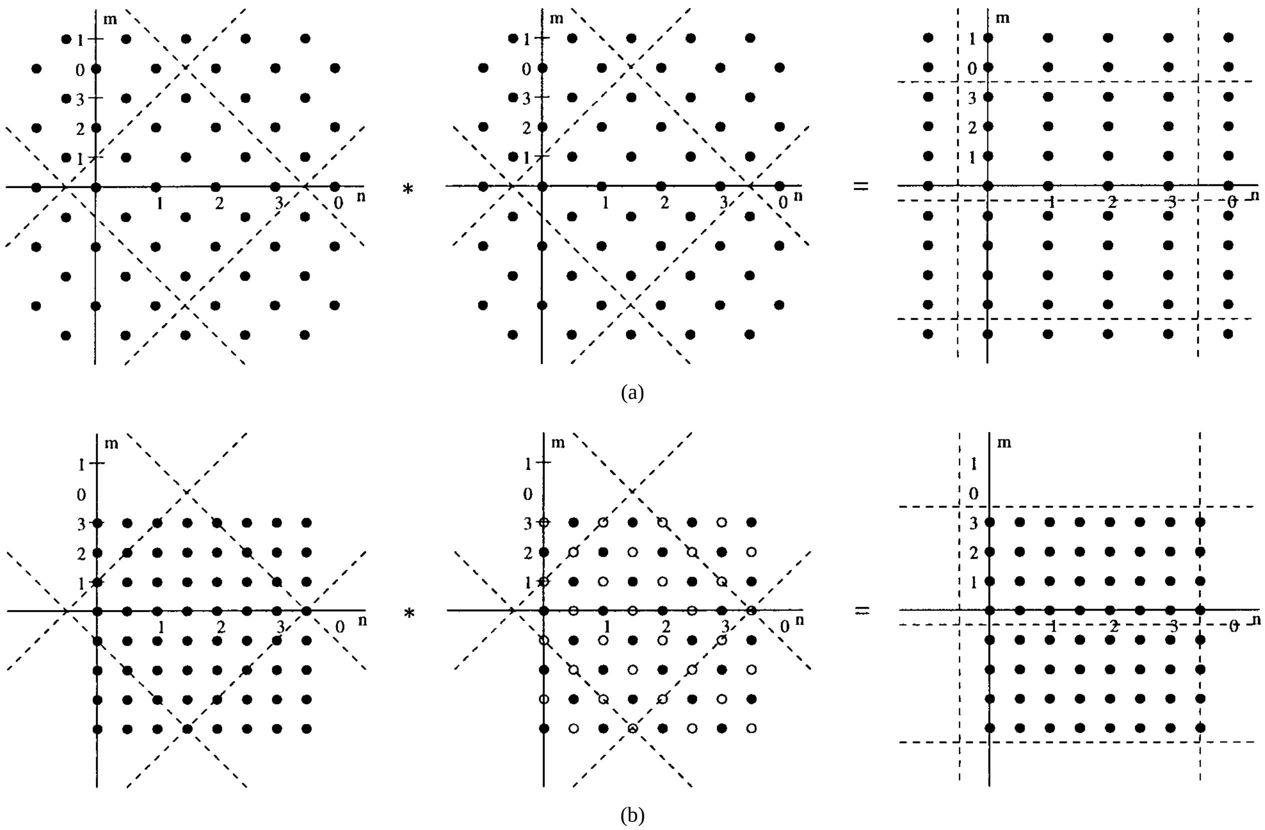


Fig. 6. Equivalent methods for computing the type IV Cohen class. On the left is the LACF, in the middle is the kernel, and on the right is the generalized LACF. The dashed lines delineate the period of each of the functions.

signals using operator theory. By extending the above proof for the type I Cohen class, we immediately generate a closed form for the entire class of quadratic, shift-covariant TFD's for type IV signals

$$\begin{aligned} C_x^{\mathrm{IV}}(n,k) &= \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} x(n_1)x^*(n_2)\psi(n_1-n,n_2-n) \\ &\quad \times e^{-j2\pi k(n_1-n_2)/N} && \text{(6a)} \\ &= \sum\sum_{0\le n'\pm\frac{m}{2}<N,m\in\mathbb{Z}} R_x^{\mathrm{IV}}(n,m)\,\phi(n-n',m)e^{-j2\pi km/N} && \text{(6b)} \\ &= \frac{1}{N}\sum_{k_1=0}^{N-1}\sum_{k_2=0}^{N-1} X(k_1)X^*(k_2)\Psi(k_1-k,k_2-k) \\ &\quad \times e^{j2\pi n(k_1-k_2)/N} && \text{(6c)} \\ &= \frac{1}{N}\sum\sum_{0\le k'\pm\frac{p}{2}<N,p\in\mathbb{Z}} R_X^{\mathrm{IV}}(p,k')\Phi(p,k-k')e^{j2\pi np/N} && \text{(6d)} \end{aligned}$$

where the type IV temporal LACF and spectral LACF are defined as

$$R_x^{\mathrm{IV}}(n,m) = x\Big(n+\frac{m}{2}\Big)x^*\Big(n-\frac{m}{2}\Big)$$
$$R_X^{\mathrm{IV}}(p,k) = X\Big(k+\frac{p}{2}\Big)X^*\Big(k-\frac{p}{2}\Big)$$

and the kernels are related to each other analogous to (3). The functions $R_x^{\mathrm{IV}}(n,m)$, $\phi(n,m)$, $R_X^{\mathrm{IV}}(p,k)$, and $\Phi(p,k)$ are all sampled on hexagonal grids and are periodic in the hexagonal sense. Examples of $R_x^{\mathrm{IV}}(n,m)$ and $\phi(n,m)$ are given in Fig. 6 for a signal of length 4.

Since the distributions produced by Richman *et al.* are shift-covariant and quadratic, they will be members of this class. The method of Narayanan *et al.* is promising but has yet to produce a closed form for the entire class. Our method is more complete than the previous two works in that we generate the entire class, and the mathematics behind our derivation are more straightforward.

The type IV Cohen class can also be considered to be a generalization of the type IV spectrogram

$$\begin{aligned} S_x^{\mathrm{IV}}(n,k) = \sum_{n_1=1}^{N}\sum_{n_2=1}^{N} & x(n_1)x^*(n_2) \\ & \times h(n_1-n)h^*(n_2-n)e^{-jk(n_1-n_2)}. \end{aligned}$$

From this, it is clear that the type IV spectrogram is a member of the type IV Cohen class and that TFD's in the type IV Cohen class can be decomposed into a weighted sum of type IV spectrograms.

A. Distributions in the Type IV Cohen Class

To convert distributions from the type I Cohen class to the type II Cohen class, we simply sampled the corresponding kernels. To convert distributions to the type IV Cohen

TABLE IV
SOME KERNELS FOR THE TYPE IV COHEN CLASS

Spectrogram:	$\phi(n,m) = h(n+\frac{m}{2})\,h^*(n-\frac{m}{2})$.
Wigner (odd N):	$\phi(n,m) = \begin{cases} \delta(n) & \text{for even } m \\ \delta(n+\frac{N}{2}) & \text{for odd } m \end{cases}$
Quasi-Wigner (even N):	$\phi(n,m) = \begin{cases} \cos^2(\pi m/2N) & \text{for } n=0 \\ \sin^2(\pi m/2N) & \text{for } n=\frac{N}{2} \\ \frac{1}{4} & \text{for } n=\{\frac{1}{2},\frac{N}{2}\pm\frac{1}{2},N-\frac{1}{2}\} \\ 0 & \text{otherwise} \end{cases}$
Rihaczek:	$\phi(n,m) = \delta(n-\frac{m}{2})$

class, we must also account for the periodic nature of the distributions. The generalization of the spectrogram and the Rihaczek distribution to type IV signals is straightforward, and the corresponding kernels are listed in Table IV. For example, the type IV Rihaczek distribution can be expressed as

$$RD_x^{\mathrm{IV}}(n,k) = x^*(n)X(k)e^{j2\pi nk/N}.$$

We have also generalized the binomial distribution to the type IV Cohen class, but we have not included the kernel in Table IV since it has a complicated form.

As mentioned above, in [36] and [37], we presented an alternative definition of the classical Wigner distribution that extends easily to the three types of discrete signals. Surprisingly, under this definition, the type IV Wigner distribution exists only for signals with an odd length period. The kernel of this distribution is listed in Table IV and is identical the definition proposed by Richman *et al.* [9]. For signals with an even length period, we have constructed a type IV quasi-Wigner distribution that performs very little smoothing and has similar characteristics to the type IV Wigner distribution.[8] This kernel is also listed in Table IV.[9]

B. Relationship with the Classical Wigner Distribution

In order to understand the properties of the type IV Cohen class, we will use a method similar to the method used for the type II Cohen class. In Fig. 6(a), we have an example of how to compute a TFD in the type IV Cohen class for a four point signal. The LACF and the kernel will be sampled on hexagonal grids and will also be periodic in the hexagonal sense. However, the convolution of the LACF with the kernel, which is called the generalized LACF, will be sampled on a rectangular grid and will also be periodic in the rectangular sense. For an N-point signal, this generalized LACF will have N-by-N points. To compute the type IV TFD, we must perform a discrete Fourier transform in the lag variable m.

In Fig. 6(b), we present an alternative method for computing TFD's in the type IV Cohen class by means of a sampled version of the classical Wigner distribution. For the first step, we will double the number of points in the LACF by performing a 2-D sinc interpolation. We will also double the number of points in the kernel by inserting zeros between all the points. The modified LACF and kernel are sampled on rectangular grids and will also be periodic in the rectangular sense. For the second step, we extract a portion of $2N$-by-$2N$ points that is exactly one period of the modified functions in the rectangular sense. This is shown pictorially in Fig. 6(b). If we denote the modified LACF as $\tilde{R}_x^{\mathrm{IV}}(n,m)$ and the modified kernel as $\tilde{\phi}(n,m)$, then the modified method pictured in Fig. 6(b) can be represented as

$$C_x^{\mathrm{IV}}(n,k) = \sum_{m=0}^{2N-1} \left[\tilde{R}_x^{\mathrm{IV}}(n,m) \circledast_n \tilde{\phi}(n,m)\right] \times e^{-j2\pi km/N}\Big|_{n,k\in[0\cdots N-1]}. \tag{7}$$

By applying Fourier transform properties, it can be shown that

$$C_x^{\mathrm{IV}}(n,k) = W_x^{\mathrm{I}}(n,k) \circledast_n \circledast_k \mathcal{P}(n,k)\big|_{n,k\in[1\cdots N]}$$

where[10]

$$\mathcal{P}(n,k) = \sum_{m=0}^{2N-1} \tilde{\phi}(n,m) + \tilde{\phi}\left(n+\frac{N}{2}, m+N\right) e^{j2\pi mk/N}$$

and $W_x^{\mathrm{I}}(n,k)$ represents samples of the classical Wigner distribution in time and frequency. For more details, see [33].

Three examples of modified kernels in the time–frequency plane $\mathcal{P}(n,k)$ are shown in Fig. 7. The first corresponds to the type IV binomial distribution, the second corresponds to the type IV quasi-Wigner distribution, and the third corresponds to a type IV spectrogram with a Hanning window. As above, all kernels satisfy $\phi(n,m) = \phi^*(-n,m)$.

Each of the kernels has four distinct parts. The first behaves as a lowpass filter both in the time and frequency directions and is similar to the kernel in the type I Cohen class. The second behaves as a lowpass filter in the frequency direction and a highpass filter in the time direction and is similar to the highpass part of a type II Cohen class kernel. The third component behaves like a lowpass filter in the time direction and a highpass filter in the frequency direction, and the fourth component behaves like a highpass filter in both the time and frequency directions.

We will now apply the above to a fictional, single-component signal with period N centered on the time–frequency torus at (n_1,k_1). Because of the four parts of the kernel, there will be three other terms centered at $(n_1+N/2,k_1)$, $(n_1,k_1+N/2)$, and $(n_1+N/2,k_1+N/2)$. Similar to the type II Cohen class, we will choose to designate these terms as cross terms between the component and the periodic repetitions of itself.

For a two-component signal, the situation is a little more complicated. Add a second component centered at (n_2,k_2). There will be three cross terms between the second component and the periodic repetitions of itself. There will also be four more cross terms occurring between the two components on the torus. These will be centered at $(\frac{n_1+n_2}{2},\frac{k_1+k_2}{2})$, $(\frac{n_1+n_2+N}{2},\frac{k_1+k_2}{2})$, $(\frac{n_1+n_2}{2},\frac{k_1+k_2+N}{2})$, $(\frac{n_1+n_2+N}{2},\frac{k_1+k_2+N}{2})$. There will be a total of 10 cross terms

[8] The type IV Wigner distribution proposed by Richman *et al.* for signals with an even length uses a different group structure than their odd length distribution and has strikingly different properties. For these reasons, it does not seem appropriate to designate their even length distribution as a "Wigner distribution."

[9] This definition arose from working backward from the modified kernels, which will be described below.

[10] Note carefully the sampling for n and m in Fig. 6.

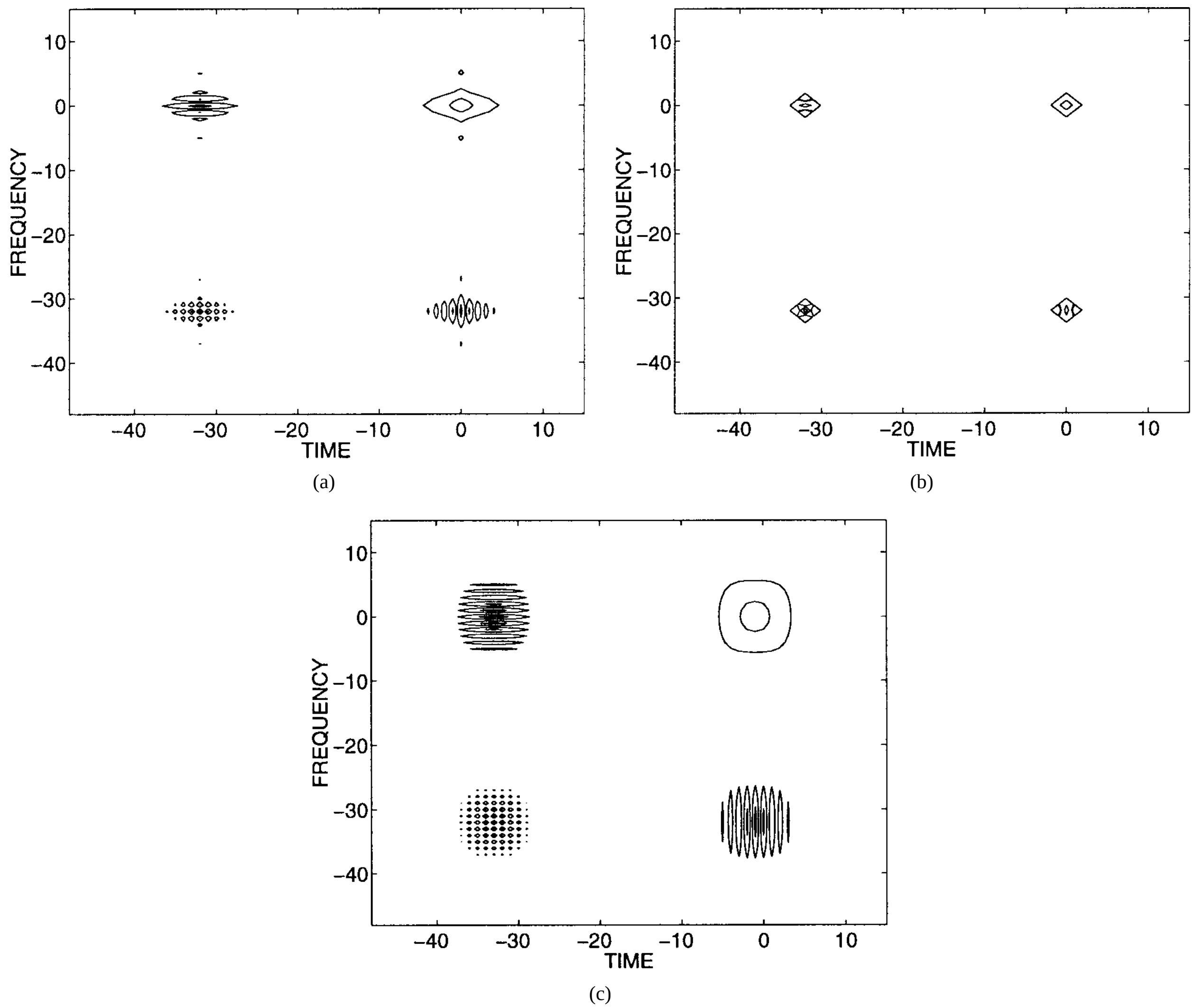


Fig. 7. Three modified kernels for the type IV Cohen class in the time-frequency plane.

for a two-component signal. The configuration of the auto terms and cross terms for a two component signal is depicted in Fig. 3(c). Note that all of these cross terms are necessary for a type IV TFD to satisfy the time and frequency shift covariance properties [33]. We will not develop the rate of oscillation of the cross terms in the time and frequency directions and the attenuation properties since they are a direct extension of the type II Cohen class outlined above.

In general, the cross terms in the type IV Cohen class satisfy the following properties, which are analogous to the properties of the type I Cohen class.[11]

- Cross terms are centered exactly between two auto terms, where between is applied on the surface of the torus.
- If two auto terms are separated in time by n_0, then the rate of oscillation of the cross term in the frequency direction will be $\min\{n_0, N - n_0\}$.
- If two auto terms are separated in frequency by k_0, then the rate of oscillation of the cross term in the time direction will be $\min\{k_0, N - k_0\}$.
- The ability of the kernel to attenuate the cross term is directly related to the distance between the cross term and the corresponding auto terms (regardless of the rate of oscillation of the cross term).

[11] Subject to the kernel constraint mentioned above.

C. Kernel Constraints and Distribution Properties

Here, we will present sufficient kernel constraints for TFD's in the type IV Cohen class to satisfy several desirable properties. In all cases, the proofs are simple extensions of those for the type I and type II Cohen classes and will be omitted. The properties and the corresponding kernel constraints are listed in Table V. The properties of finite time support and finite frequency support are not well defined since type IV TFD's are periodic in both time and frequency; however, the strong time support and strong frequency support properties can be satisfied. An example of a TFD that satisfies the Moyal formula, the strong time support property, and the strong frequency support property is the type IV Rihaczek distribution.

For the instantaneous frequency property, we assumed a signal of the form $x(n) = e^{j\varphi(n)}$, and for the group delay property, we assumed a signal of the form $X(k) = e^{j\varphi(k)}$. Other alternatives for the instantaneous frequency and group delay properties could have been considered, as was done for the type II Cohen class in [33].

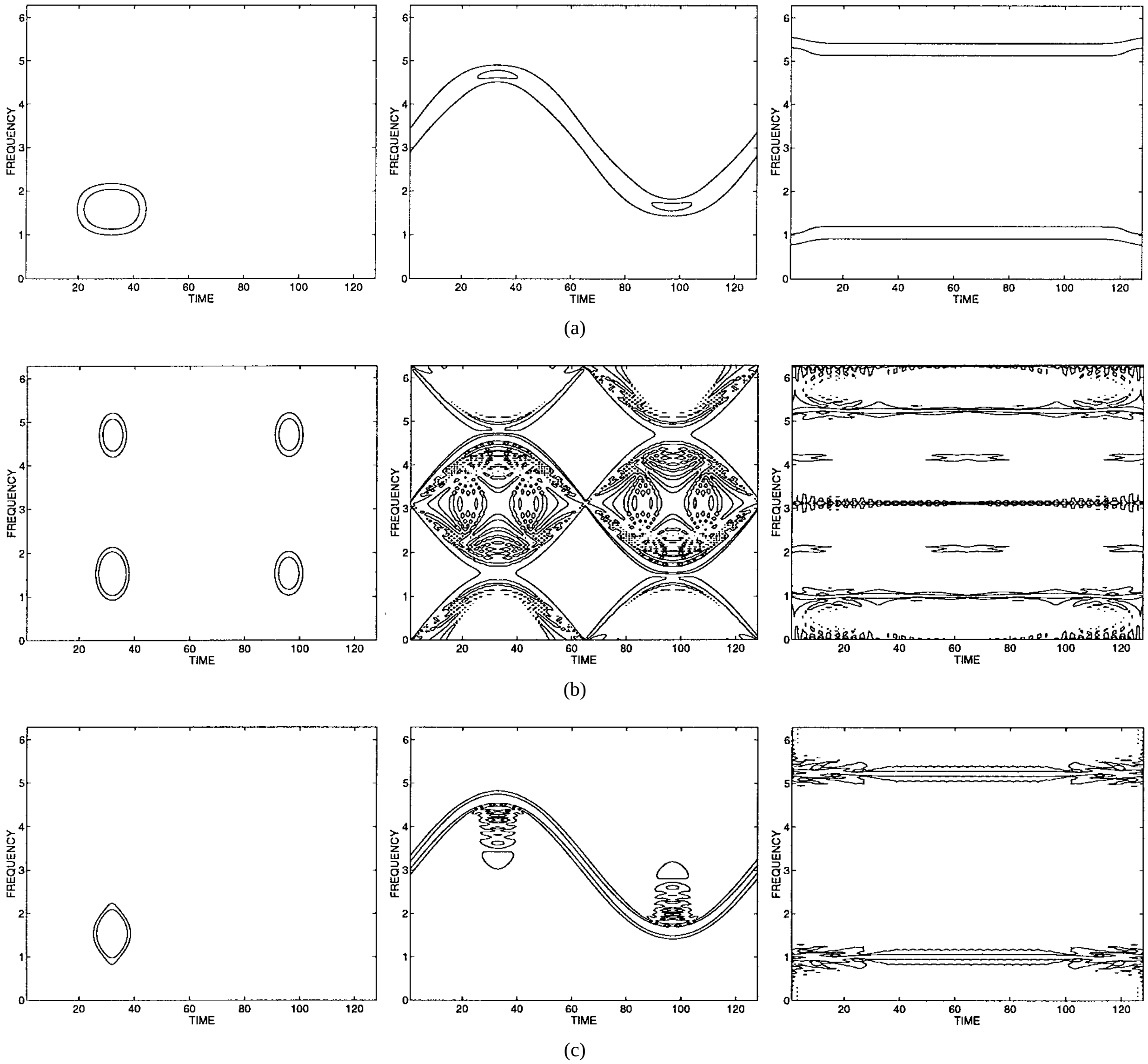


Fig. 8. Examples of TFD's in the type IV Cohen class. On the left is a single Gabor logon, in the middle is a signal with a sinusoidal instantaneous frequency, and on the right is an aperiodic sinusoid.

D. Examples

We now prevent several examples to illustrate the properties of the type IV Cohen class. We will use three of the type IV distributions mentioned above. The first is a type IV spectrogram with a Hanning window. Like the type I spectrogram, this distribution eliminates cross terms very well but also suffers from poor resolution. The second is the quasi-Wigner distribution, which provides very little smoothing and is useful for displaying the properties of the cross terms in the type IV Cohen class. The third is the type IV binomial distribution, which provides for a tradeoff between the first two in terms of maintaining resolution and suppressing cross terms. In Fig. 8, we present type IV TFD's corresponding to these kernels for three different test signals.

The first signal is a Gabor logon and is defined as

$$x(n) = e^{-(n-32)^2/20} e^{j\pi n/4} \quad n = 1 \cdots 128.$$

The quasi-Wigner distribution clearly shows the cross terms as predicted above. The cross term in the upper-right corner has a negative amplitude and cancels out the other cross terms in the marginal calculations. The spectrogram and binomial TFD attenuate the cross terms and provide some smoothing of the auto term.

The second signal has a sinusoidal IF and is defined as

$$x(n) = e^{j(\pi n - 32\cos(2\pi n/128))} \quad n = 1 \cdots 128.$$

Notice that none of the TFD's show edge effects that would be apparent in a type II TFD. The three TFD's illustrate the tradeoff between resolution and cross term suppression.

The third signal is an aperiodic sinusoid

$$x(n) = \sin(\pi n/3) \quad n = 1 \cdots 128$$

By aperiodic, we mean that since 3 is not a divisor of 128, the signal is not continuous at the period boundary. The magnitude of the DFT of this signal contains "leakage," and this also

TABLE V
KERNEL CONSTRAINTS FOR THE TYPE IV COHEN CLASS

Real:	$\psi(n_1,n_2)=\psi^*(n_2,n_1) \quad \Longrightarrow \quad \Im\{C_x^{IV}(n,k)\}=0$
Positive:	$\psi(n_1,n_2)=h(n_1)\,h^*(n_2) \quad \Longrightarrow \quad C_x^{IV}(n,k)\geq 0$
Time Marginal:	$\psi(n,n)=\delta(n)/N \quad \Longrightarrow \quad \sum_{k=1}^{N} C_x^{IV}(n,k)=\lvert x(n)\rvert^2$
Frequency Marginal:	$\sum_{n=1}^{N}\psi(n,n+c)=1 \quad \forall c \quad \Longrightarrow \quad \sum_{n=1}^{N} C_x^{IV}(n,k)=\lvert X(k)\rvert^2$
Strong Time Support:	$\psi(n_1,n_2)=0 \quad \text{for} \quad \lvert n_1\rvert\neq\lvert n_2\rvert \quad \Longrightarrow$ $C_x^{IV}(n,k)=0 \quad \forall n \quad \text{such that} \quad x(n)=0$
Strong Frequency Support:	$\Psi(k_1,k_2)=0 \quad \text{for} \quad \lvert k_1\rvert\neq\lvert k_2\rvert \quad \Longrightarrow$ $C_x^{IV}(n,k)=0 \quad \forall k \quad \text{such that} \quad X(k)=0$
The Moyal Formula:	$\sum_{n=1}^{N}\psi(n+c,n)\,\psi^*(n+c+a,n+a)=\delta(a) \quad \forall c \quad \Longrightarrow$ $\sum_{n,k=1}^{N} C_x^{IV}(n,k)\left[C_y^{IV}(n,k)\right]^*=\left\lvert\sum_{n=1}^{N} x(n)\,y^*(n)\right\rvert^2$
Instantaneous Frequency:	$\psi(n,n-1)=\delta(n)+\delta(n-1) \quad \Longrightarrow$ $\arg\sum_{k=1}^{N} e^{j2\pi n/N}\,C_x^{IV}(n,k)=\left[\frac{N}{2\pi}\,\frac{\varphi(n+1)-\varphi(n-1)}{2}\right] \mod N$
Group Delay:	$\Psi(k,k-1)=\delta(k)+\delta(k-1) \quad \Longrightarrow$ $\arg\sum_{n=1}^{N} e^{j2\pi k/N}\,C_x^{IV}(n,k)=\left[\frac{N}{2\pi}\,\frac{\varphi(k+1)-\varphi(k-1)}{2}\right] \mod N$

occurs in the type IV TFD's. All three of the TFD's show a discontinuity at the period edge. The quasi-Wigner shows some energy at a frequencies of $2\pi/3$ and $4\pi/3$ rad that is a result of the discontinuity but does not have an obvious interpretation.

VII. PRACTICAL ISSUES

Given that the type I Cohen class has fewer cross terms than the discrete Cohen classes, we might wonder why the discrete Cohen classes are necessary. Here, we give three advantages of the discrete Cohen classes over the original Cohen class and present a comparison with another discrete class.

First, to compute distributions in the type I Cohen class, we must use a signal that is oversampled by a factor of two, which increases by a factor of four the required computations. Second, even with oversampling, distributions in the type I Cohen class are not always straightforward to compute. For example, the Choï–Williams distribution [45] is often cited in the literature, but the kernel is not bandlimited and does not have compact support. As a result, it is not clear how to sample the kernel and, thus, provide an accurate implementation of this distribution. Third, the discrete Cohen classes provide the framework for relating discrete TFD's to other discrete-time processing such as linear, time-varying filtering [46], and signal detection [41].

Boashash [13] has created a class of discrete TFD's for type II signals called the generalized discrete time–frequency distributions (GDTFD's). While the implementation of this method is slightly simpler conceptually since all functions are defined on rectangular grids, there are many disadvantages to this method. The GDTFD's have the following disadvantages.

- They do not compute samples of the type I Cohen class.
- They are covariant to neither linear nor circular frequency shifts.
- They do not include any type of spectrogram.
- They require an oversampled signal and are thus more expensive computationally.

The GDTFD's and their relationships with the type I and type II Cohen classes are examined in much greater detail in [33]. Given the above, the Cohen classes seem preferable to the GDTFD's.

VIII. CONCLUSION

The most well known and most often used TFD's are those in the Cohen class. The Cohen class is often called the shift-covariant class because it can be shown to include every quadratic TFD that is covariant to time shifts and frequency shifts. In this paper, we have used the axioms of shift covariance to extend the original Cohen class to the three types of discrete signals in Table I. The extension is relatively simple and immediately generates a closed form for the entire class.

Having a closed form for the classes does not immediately provide an understanding of the properties of the classes. To this end, we have provided an explicit relationship between the classical Wigner distribution (whose properties are very well known) and the discrete Cohen classes. With this relationship, it is straightforward to see that the properties of the discrete Cohen classes, while different from the original Cohen class, are a direct consequence of the periodicities of the discrete signals.

All TFD's mentioned in this paper can be computed from a MATLAB software package that is freely available at http://mdsp.bu.edu/jeffo.

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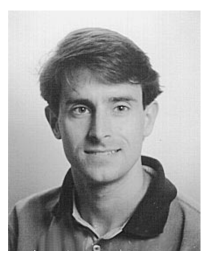

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